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Mathieu Functions of Integral Order and Their Tabulation

Introduction

Increasing attention is being given to scientific and technical problems which lead to differential equations of the Mathieu type. A considerable amount of analytical knowledge concerning the solutions of such equations exists. To a physicist or an engineer, however, a formal solution is seldom sufficient, but is merely a stepping-stone to quantitative results. There is thus an increasing need for tables giving numerical values of solutions which are of use in applications.

In a previous article¹ one of us gave an account of such tables as already exist. The aims of the present article are (a) to define a set of Mathieu functions appropriate to the solution of potential, wave, and analogous problems, (b) to make suggestions as to which should be tabulated, (c) to indicate which formulae are most convenient for computation.

Use of a common notation and agreement as to which functions are to be tabulated would greatly accelerate progress. The notation and definitions which we employ are a natural and systematic extension of those finally adopted² by INCE in his "Tables"³ and embody the results of much thought and discussion. The 'e' occurring throughout is of course 'e for elliptic.'

Differential equations

If the two-dimensional wave equation⁴

$$(1) \quad \nabla^2 \phi + \kappa^2 \phi = 0$$

is transformed from rectangular co-ordinates (x, y) to elliptic co-ordinates (ξ, η) by the formulae

$$(2) \quad x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta,$$

and a solution of the form

$$(3) \quad \phi = X(\xi)Y(\eta)$$

is sought, it is found that $X(\xi)$ and $Y(\eta)$ must satisfy respectively the equations

$$(4) \quad d^2 Y/d\eta^2 + (a - 2q \cos 2\eta)Y = 0,$$

and

$$(5) \quad d^2 X/d\xi^2 - (a - 2q \cosh 2\xi)X = 0,$$

in which

$$(6) \quad q = \frac{1}{4}\kappa^2 c^2,$$

and a is the separation constant.

2 MATHIEU FUNCTIONS OF INTEGRAL ORDER AND THEIR TABULATION

Equation (4) is known as Mathieu's equation. We adopt, as the canonical form,

$$(7) \quad d^2y/dz^2 + (a - 2q \cos 2z)y = 0.$$

Corresponding to (5) we write

$$(8) \quad d^2y/dz^2 - (a - 2q \cosh 2z)y = 0.$$

It is a very fortunate circumstance that (7) transforms into (8), and vice versa, if z is replaced by $\pm iz$.

In other problems the sign of κ^2 in (1) is changed, and we are then led to the pair of equations

$$(9) \quad d^2y/dz^2 + (a + 2q \cos 2z)y = 0,$$

$$(10) \quad d^2y/dz^2 - (a + 2q \cosh 2z)y = 0,$$

which, again, are interchanged by replacing z by $\pm iz$.

We note also that (7) is transformed into (9), and vice versa, by replacing z by $\pm (\frac{1}{2}\pi \pm z)$, and (8) into (10) and vice versa by replacing z by $\pm (\frac{1}{2}i\pi \pm z)$.

Physical considerations are usually such that solutions of (7) or (9) must admit the period 2π in z . Such solutions we shall term, for brevity, 'periodic solutions,' although solutions of period $2s\pi$, where s is any integer, exist. The present article deals mainly with periodic solutions of (7) or (9) and with the corresponding solutions of (8) and (10); usually q will be regarded as real and positive. We shall frequently write

$$(11) \quad q = k^2,$$

and it is clear from (6) that k , rather than q , is likely to be physically significant.

Periodic solutions of (7)

In order that a solution of (7), with a prescribed value of q , may be periodic, the separation constant a must be one of an infinite sequence of characteristic numbers. The corresponding solutions fall into four classes, according to their symmetry or anti-symmetry, about $z = 0$ and $z = \frac{1}{2}\pi$. Following HEINE,⁶ and using the *ce*, *se* notation⁶ suggested by WHITTAKER, Ince⁸ defines these as⁷

$$(12.1) \quad ce_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz, \quad (a_{2n}),$$

$$(12.2) \quad se_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin (2r+1)z, \quad (b_{2n+1}),$$

$$(12.3) \quad ce_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos (2r+1)z, \quad (a_{2n+1}),$$

$$(12.4) \quad se_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin (2r+2)z, \quad (b_{2n+2}).$$

The symbols in parentheses denote the corresponding characteristic numbers. As listed, for a given positive q , these are in order of increasing magnitude. In future formulae, the appropriate characteristic number can be inferred immediately from the coefficients and the order of the functions. In the above, n may be zero or a positive integer; the corresponding function has n real zeros in the open interval $0 < z < \frac{1}{2}\pi$.

Recurrence relations between the coefficients, derived by substituting these series in the differential equation (7), lead to a technique (developed by GOLDSTEIN² and Ince,³ and now well-known), for the determination of characteristic numbers and the ratios of successive coefficients. The latter are made precise by the normalization rule

$$(13) \quad \int_0^{2\pi} y^2 dz = \pi,$$

so that

$$(14) \quad 1 = 2A_0^2 + \sum_{r=1}^{\infty} A_{2r}^2 = \sum_{r=0}^{\infty} B_{2r+1}^2 = \sum_{r=0}^{\infty} A_{2r+1}^2 = \sum_{r=0}^{\infty} B_{2r+2}^2.$$

The functions so defined may be termed elliptic cylinder functions, or Mathieu functions, of (zero and) positive integral order. They have been tabulated by Ince³ for $n = 0, 1, 2$, $q = 1(1)10$, $0 \leq z \leq \frac{1}{2}\pi$.

As $q \rightarrow 0$, $ce_0(z, q) \rightarrow 1/\sqrt{2}$,

$$ce_m(z, q) \rightarrow \cos mz, \quad se_m(z, q) \rightarrow \sin mz \quad (m > 0),$$

and as regards symmetry or antisymmetry about $z = 0$ or $z = \frac{1}{2}\pi$, the functions behave as do these limiting forms; it is therefore sufficient to tabulate the functions over the first quadrant.

The series (12.1)–(12.4) are absolutely and uniformly convergent for all finite (real or complex) values of z . For small or moderate values of q and n , and for real values of z , convergence is satisfactorily rapid for computation.

Periodic solutions of (9)

It has already been mentioned that (9) may be obtained from (7) by replacing z by $\pm (\frac{1}{2}\pi \pm z)$. Hence if $f(z, q)$ satisfies (7), then $f(\frac{1}{2}\pi - z, q)$ satisfies (9).

Consequently we define (following Ince³)

$$(15.1) \quad ce_{2n}(z, -q) = (-)^n ce_{2n}(\frac{1}{2}\pi - z, q) = (-)^n \sum (-)^r A_{2r} \cos 2rz,$$

$$(15.2) \quad ce_{2n+1}(z, -q) = (-)^n se_{2n+1}(\frac{1}{2}\pi - z, q) \\ = (-)^n \sum (-)^r B_{2r+1} \cos (2r+1)z,$$

$$(15.3) \quad se_{2n+1}(z, -q) = (-)^n ce_{2n+1}(\frac{1}{2}\pi - z, q) \\ = (-)^n \sum (-)^r A_{2r+1} \sin (2r+1)z,$$

$$(15.4) \quad se_{2n+2}(z, -q) = (-)^n se_{2n+2}(\frac{1}{2}\pi - z, q) \\ = (-)^n \sum (-)^r B_{2r+2} \sin (2r+2)z.$$

The signs have been chosen so that the limiting forms as $q \rightarrow 0$ are $+1/\sqrt{2}$, $+\cos mz$, or $+\sin mz$, as the case may be.

It is clear that tables of the periodic solutions of (7) will also yield the values of the corresponding periodic solutions of (9).

Second solutions of (7) and (9)

It is known that two linearly independent solutions of (7) or (9), both admitting the period 2π , cannot coexist for the same values of a and q ($\neq 0$).

The second solution corresponding to a periodic Mathieu function of integral order is therefore not periodic.

Second solutions corresponding to $ce_m(z, q)$ and $se_m(z, q)$ will be denoted by $fe_m(z, q)$ and $ge_m(z, q)$, respectively.¹⁰ Not being periodic, they will rarely be of importance in applications. Goldstein¹¹ has given (unnormalized) definitions, and an elaborate method for computing the coefficients. A much less laborious method of computation has been discovered, and it is hoped that an account of this will be published shortly.

First solution of (8)

Since (7) is transformed into (8) by writing iz for z , we can derive a set of solutions of (8) by applying this transformation to (12.1)–(12.4): of these

$$Ce_{2n}(z, q) = ce_{2n}(iz, q) = \sum A_{2r} \cosh 2rz$$

is typical. The rate of convergence of such series decreases rapidly with increasing (real) z . For large z they are consequently inconvenient for computation. Moreover, asymptotic properties cannot readily be derived from them. Fortunately, other developments are known, involving Bessel functions, which do not have these defects. The set of formulae, in which $k = +\sqrt{q}$ and primes denote derivatives, is:

$$(16.11) \quad Ce_{2n}(z, q) = ce_{2n}(iz, q) = \sum A_{2r} \cosh 2rz$$

$$(16.12) \quad = \frac{ce_{2n}(\frac{1}{2}\pi, q)}{A_0} \sum (-)^r A_{2r} J_{2r}(2k \cosh z)$$

$$(16.13) \quad = \frac{ce_{2n}(0, q)}{A_0} \sum A_{2r} J_{2r}(2k \sinh z)$$

$$(16.14) \quad = \frac{ce_{2n}(0, q) ce_{2n}(\frac{1}{2}\pi, q)}{A_0^2} \sum (-)^r A_{2r} J_r(ke^{-z}) J_r(ke^z).$$

$$(16.21) \quad Se_{2n+1}(z, q) = i^{-1} se_{2n+1}(iz, q) = \sum B_{2r+1} \sinh (2r+1)z$$

$$(16.22) \quad = \frac{se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1} \tanh z \sum (-)^r (2r+1) B_{2r+1} J_{2r+1}(2k \cosh z)$$

$$(16.23) \quad = \frac{se'_{2n+1}(0, q)}{kB_1} \sum B_{2r+1} J_{2r+1}(2k \sinh z)$$

$$(16.24) \quad = \frac{se'_{2n+1}(0, q) se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1^2} \sum (-)^r B_{2r+1} \\ \times \{J_r(ke^{-z}) J_{r+1}(ke^z) - J_{r+1}(ke^{-z}) J_r(ke^z)\}.$$

$$(16.31) \quad Ce_{2n+1}(z, q) = ce_{2n+1}(iz, q) = \sum A_{2r+1} \cosh (2r+1)z$$

$$(16.32) \quad = -\frac{ce'_{2n+1}(\frac{1}{2}\pi, q)}{kA_1} \sum (-)^r A_{2r+1} J_{2r+1}(2k \cosh z)$$

$$(16.33) \quad = \frac{ce_{2n+1}(0, q)}{kA_1} \coth z \sum (2r+1) A_{2r+1} J_{2r+1}(2k \sinh z)$$

$$(16.34) \quad = -\frac{ce_{2n+1}(0, q)ce'_{2n+1}(\frac{1}{2}\pi, q)}{kA_1^2} \sum (-)^r A_{2r+1} \\ \times \{J_r(ke^{-z})J_{r+1}(ke^z) + J_{r+1}(ke^{-z})J_r(ke^z)\}.$$

$$(16.41) \quad Se_{2n+2}(z, q) = i^{-1}se_{2n+2}(iz, q) = \sum B_{2r+2} \sinh (2r+2)z$$

$$(16.42) \quad = -\frac{se'_{2n+2}(\frac{1}{2}\pi, q)}{k^2 B_2} \tanh z \sum (-)^r (2r+2) B_{2r+2} J_{2r+2}(2k \cosh z)$$

$$(16.43) \quad = \frac{se'_{2n+2}(0, q)}{k^2 B_2} \coth z \sum (2r+2) B_{2r+2} J_{2r+2}(2k \sinh z)$$

$$(16.44) \quad = -\frac{se'_{2n+2}(0, q)se'_{2n+2}(\frac{1}{2}\pi, q)}{k^2 B_2^2} \sum (-)^r B_{2r+2} \\ \times \{J_r(ke^{-z})J_{r+2}(ke^z) - J_{r+2}(ke^{-z})J_r(ke^z)\}.$$

The series involving Bessel functions converge absolutely and uniformly for all finite values of z . The rate of convergence of those involving products of Bessel functions increases with increasing z .

For large real values of z these functions oscillate with increasing rapidity, while the amplitude tends exponentially to zero.

As $q \rightarrow 0$, $Ce_0(z, q) \rightarrow 1/\sqrt{2}$, and, if $m > 0$, $Ce_m(z, q) \rightarrow \cosh mz$, $Se_m(z, q) \rightarrow \sinh mz$.

First solutions of (10)

A similar set of solutions of (10) exists, and can be obtained by writing $(\frac{1}{2}i\pi + z)$ for z in (16.11)–(16.44).

$$(17.11) \quad Ce_{2n}(z, -q) = (-)^n \sum (-)^r A_{2r} \cosh 2rz$$

$$(17.12) \quad = (-)^n \frac{ce_{2n}(\frac{1}{2}\pi, q)}{A_0} \sum A_{2r} I_{2r}(2k \sinh z)$$

$$(17.13) \quad = (-)^n \frac{ce_{2n}(0, q)}{A_0} \sum (-)^r A_{2r} I_{2r}(2k \cosh z)$$

$$(17.14) \quad = (-)^n \frac{ce_{2n}(0, q)ce_{2n}(\frac{1}{2}\pi, q)}{A_0^2} \sum (-)^r A_{2r} I_r(ke^{-z}) I_r(ke^z).$$

$$(17.21) \quad Ce_{2n+1}(z, -q) = (-)^n \sum (-)^r B_{2r+1} \cosh (2r+1)z$$

$$(17.22) \quad = (-)^n \frac{se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1} \coth z \sum (2r+1) B_{2r+1} I_{2r+1}(2k \sinh z)$$

$$(17.23) \quad = (-)^n \frac{se'_{2n+1}(0, q)}{kB_1} \sum (-)^r B_{2r+1} I_{2r+1}(2k \cosh z)$$

$$(17.24) \quad = (-)^n \frac{se'_{2n+1}(0, q)se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1^2} \sum (-)^r B_{2r+1} \\ \times \{I_r(ke^{-z}) I_{r+1}(ke^z) + I_{r+1}(ke^{-z}) I_r(ke^z)\}.$$

$$(17.31) \quad Se_{2n+1}(z, -q) = (-)^n \sum (-)^r A_{2r+1} \sinh(2r+1)z$$

$$(17.32) \quad = (-)^{n+1} \frac{ce'_{2n+1}(\frac{1}{2}\pi, q)}{kA_1} \sum A_{2r+1} I_{2r+1}(2k \sinh z)$$

$$(17.33) \quad = (-)^n \frac{ce_{2n+1}(0, q)}{kA_1} \tanh z \\ \times \sum (-)^r (2r+1) A_{2r+1} I_{2r+1}(2k \cosh z)$$

$$(17.34) \quad = (-)^n \frac{ce_{2n+1}(0, q) ce'_{2n+1}(\frac{1}{2}\pi, q)}{kA_1^2} \sum (-)^r A_{2r+1} \\ \times \{I_r(ke^{-z}) I_{r+1}(ke^z) - I_{r+1}(ke^{-z}) I_r(ke^z)\}.$$

$$(17.41) \quad Se_{2n+2}(z, -q) = (-)^n \sum (-)^r B_{2r+2} \sinh(2r+2)z$$

$$(17.42) \quad = (-)^{n+1} \frac{se'_{2n+2}(\frac{1}{2}\pi, q)}{k^2 B_2} \coth z \\ \times \sum (2r+2) B_{2r+2} I_{2r+2}(2k \sinh z)$$

$$(17.43) \quad = (-)^n \frac{se'_{2n+2}(0, q)}{k^2 B_2} \tanh z \\ \times \sum (-)^r (2r+2) B_{2r+2} I_{2r+2}(2k \cosh z)$$

$$(17.44) \quad = (-)^{n+1} \frac{se'_{2n+2}(0, q) se'_{2n+2}(\frac{1}{2}\pi, q)}{k^2 B_2^2} \sum (-)^r B_{2r+2} \\ \times \{I_r(ke^{-z}) I_{r+2}(ke^z) - I_{r+2}(ke^{-z}) I_r(ke^z)\}.$$

As regards convergence, these series behave as do the corresponding ones of (16.11)–(16.44). The sign has been so chosen that as $q \rightarrow 0$ they tend to $+1/\sqrt{2}$, $+\cosh mz$, or $+\sinh mz$, as the case may be.

For large (real) values of z they tend exponentially to infinity.

Second solutions of (8)

Since the J and Y Bessel functions satisfy the same differential equations and recurrence formulae, and since it is by virtue of these that the Bessel function series in (16.11)–(16.44) satisfy (8), it follows that solutions of (8), linearly independent of (16.11)–(16.44), can be obtained by replacing therein J by Y . Of the resulting series, those involving $Y_m(2k \cosh z)$ converge only when $|\cosh z| > 1$, those involving $Y_m(2k \sinh z)$ only when $|\sinh z| > 1$. Those containing $J_m(ke^{-z}) Y_p(ke^z)$ are useful *only when the real part of z is positive*, the ratio of successive terms tending to $k^2 e^{2z}/4r^2$; those containing $J_m(ke^z) Y_p(ke^{-z})$ *only when the real part of z is negative*, the ratio of successive terms tending to $k^2 e^{2z}/4r^2$; and those containing $Y_m(ke^{-z}) Y_p(ke^z)$ not at all, the ratio of successive terms tending to unity. There remain

$$(18.11) \quad Fey_{2n}(z, q) = \frac{ce_{2n}(\frac{1}{2}\pi, q)}{A_0} \sum (-)^r A_{2r} Y_{2r}(2k \cosh z) \quad |\cosh z| > 1$$

$$(18.12) \quad = \frac{ce_{2n}(0, q)}{A_0} \sum A_{2r} Y_{2r}(2k \sinh z) \quad |\sinh z| > 1, \operatorname{Re}(z) > 0$$

$$(18.13) \quad = \frac{ce_{2n}(0, q) ce_{2n}(\frac{1}{2}\pi, q)}{A_0^2} \sum (-)^r A_{2r} J_r(ke^{-z}) Y_r(ke^z).$$

$$(18.21) \quad \text{Gey}_{2n+1}(z, q) = \frac{se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1} \tanh z \\ \times \sum (-)^r (2r+1) B_{2r+1} Y_{2r+1}(2k \cosh z) \quad |\cosh z| > 1$$

$$(18.22) \quad = \frac{se'_{2n+1}(0, q)}{kB_1} \sum B_{2r+1} Y_{2r+1}(2k \sinh z) \\ |\sinh z| > 1, \operatorname{Re}(z) > 0$$

$$(18.23) \quad = \frac{se'_{2n+1}(0, q) se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1^2} \sum (-)^r B_{2r+1} \\ \times \{J_r(ke^{-z}) Y_{r+1}(ke^z) - J_{r+1}(ke^{-z}) Y_r(ke^z)\}.$$

$$(18.31) \quad \text{Fey}_{2n+1}(z, q) = -\frac{ce'_{2n+1}(\frac{1}{2}\pi, q)}{kA_1} \sum (-)^r A_{2r+1} Y_{2r+1}(2k \cosh z) \\ |\cosh z| > 1$$

$$(18.32) \quad = \frac{ce_{2n+1}(0, q)}{kA_1} \coth z \sum (2r+1) A_{2r+1} Y_{2r+1}(2k \sinh z) \\ |\sinh z| > 1, \operatorname{Re}(z) > 0$$

$$(18.33) \quad = -\frac{ce_{2n+1}(0, q) ce'_{2n+1}(\frac{1}{2}\pi, q)}{kA_1^2} \sum (-)^r A_{2r+1} \\ \times \{J_r(ke^{-z}) Y_{r+1}(ke^z) + J_{r+1}(ke^{-z}) Y_r(ke^z)\}.$$

$$(18.41) \quad \text{Gey}_{2n+2}(z, q) = -\frac{se'_{2n+2}(\frac{1}{2}\pi, q)}{k^2 B_2} \tanh z \\ \times \sum (-)^r (2r+2) B_{2r+2} Y_{2r+2}(2k \cosh z) \quad |\cosh z| > 1$$

$$(18.42) \quad = \frac{se'_{2n+2}(0, q)}{k^2 B_2} \coth z \sum (2r+2) B_{2r+2} Y_{2r+2}(2k \sinh z) \\ |\sinh z| > 1, \operatorname{Re}(z) > 0$$

$$(18.43) \quad = -\frac{se'_{2n+2}(0, q) se'_{2n+2}(\frac{1}{2}\pi, q)}{k^2 B_2^2} \sum (-)^r B_{2r+2} \\ \times \{J_r(ke^{-z}) Y_{r+2}(ke^z) - J_{r+2}(ke^{-z}) Y_r(ke^z)\}.$$

The functions here defined and their derivations remain finite as $z \rightarrow 0$, but the series involving functions with arguments $\cosh z$ or $\sinh z$ converge non-uniformly as $|\cosh z| \rightarrow 1$ or $|\sinh z| \rightarrow 1$, so that term-by-term differentiation of these series is not legitimate at the limits of convergence.

For computational purposes the series involving Bessel function products behave admirably for all finite real positive values of z (compare the numerical example below).

For large (real) values of z the functions oscillate in the same manner as the corresponding $C_m(z, q)$ or $S_m(z, q)$, with a phase difference of $\frac{1}{2}\pi$.

Solutions of (8) exist, derived from $fe_m(z, q)$ and $ge_m(z, q)$ by replacing z by iz ; we shall denote them by $Fe_m(z, q)$ and $Ge_m(z, q)$. In applications, however, the functions $Fey_m(z, q)$ and $Gey_m(z, q)$ are much more convenient—and this is true quite apart from the slow convergence of the series defining $Fe_m(z, q)$ and $Ge_m(z, q)$ for large (real) values of z .

Second solutions of (10)

These may be obtained from the Bessel function series in (17.12)–(17.44) by replacing I_m by $(-)^m K_m$. As regards convergence, the resulting

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series behave as do the corresponding series in (18.11)–(18.43). We introduce a factor $1/\pi$ to systematize the asymptotic relations, and define:

$$(19.11) \quad Fek_{2n}(z, -q) = (-)^n \frac{ce_{2n}(\frac{1}{2}\pi, q)}{\pi A_0} \sum A_{2r} K_{2r}(2k \sinh z) \quad |\sinh z| > 1, \operatorname{Re}(z) > 0$$

$$(19.12) \quad = (-)^n \frac{ce_{2n}(0, q)}{\pi A_0} \sum (-)^r A_{2r} K_{2r}(2k \cosh z) \quad |\cosh z| > 1$$

$$(19.13) \quad = (-)^n \frac{ce_{2n}(0, q) ce_{2n}(\frac{1}{2}\pi, q)}{\pi A_0^2} \sum A_{2r} I_r(ke^{-z}) K_r(ke^z).$$

$$(19.21) \quad Fek_{2n+1}(z, -q) = \frac{se_{2n+1}(\frac{1}{2}\pi, q)}{\pi k B_1} \coth z \times \sum (2r+1) B_{2r+1} K_{2r+1}(2k \sinh z) \quad |\sinh z| > 1, \operatorname{Re}(z) > 0$$

$$(19.22) \quad = (-)^n \frac{se'_{2n+1}(0, q)}{\pi k B_1} \sum (-)^r B_{2r+1} K_{2r+1}(2k \cosh z) \quad |\cosh z| > 1$$

$$(19.23) \quad = (-)^n \frac{se'_{2n+1}(0, q) se_{2n+1}(\frac{1}{2}\pi, q)}{\pi k B_1^2} \sum B_{2r+1} \times \{I_r(ke^{-z}) K_{r+1}(ke^z) - I_{r+1}(ke^{-z}) K_r(ke^z)\}.$$

$$(19.31) \quad Gek_{2n+1}(z, -q) = (-)^{n+1} \frac{ce'_{2n+1}(\frac{1}{2}\pi, q)}{\pi k A_1} \sum A_{2r+1} K_{2r+1}(2k \sinh z) \quad |\sinh z| > 1, \operatorname{Re}(z) > 0$$

$$(19.32) \quad = (-)^n \frac{ce_{2n+1}(0, q)}{\pi k A_1} \tanh z \times \sum (-)^r (2r+1) A_{2r+1} K_{2r+1}(2k \cosh z) \quad |\cosh z| > 1$$

$$(19.33) \quad = (-)^{n+1} \frac{ce_{2n+1}(0, q) ce'_{2n+1}(\frac{1}{2}\pi, q)}{\pi k A_1^2} \sum A_{2r+1} \times \{I_r(ke^{-z}) K_{r+1}(ke^z) + I_{r+1}(ke^{-z}) K_r(ke^z)\}.$$

$$(19.41) \quad Gek_{2n+2}(z, -q) = (-)^{n+1} \frac{se'_{2n+2}(\frac{1}{2}\pi, q)}{\pi k^2 B_2} \coth z \times \sum (2r+2) B_{2r+2} K_{2r+2}(2k \sinh z) \quad |\sinh z| > 1, \operatorname{Re}(z) > 0$$

$$(19.42) \quad = (-)^n \frac{se'_{2n+2}(0, q)}{\pi k^2 B_2} \tanh z \times \sum (-)^r (2r+2) B_{2r+2} K_{2r+2}(2k \cosh z) \quad |\cosh z| > 1$$

$$(19.43) \quad = (-)^{n+1} \frac{se'_{2n+2}(0, q) se'_{2n+2}(\frac{1}{2}\pi, q)}{\pi k^2 B_2^2} \sum B_{2r+2} \times \{I_r(ke^{-z}) K_{r+2}(ke^z) - I_{r+2}(ke^{-z}) K_r(ke^z)\}.$$

These functions behave, for small values of z , in a manner similar to the corresponding Fey or Gey function. For large (real) values of z they tend exponentially to zero.

The numerical behavior of these series, and, in particular, the superiority of the Bessel function product series for computation will be demonstrated later.

Since

$$(20) \quad K_r(x) = \frac{1}{2}\pi e^{\frac{1}{2}(r+1)\pi i} \{J_r(ix) + iY_r(ix)\}$$

there are relations of the type

$$(21) \quad Fey_{2n}(z, -q) = iC_{2n}(z, -q) - 2Fek_{2n}(z, -q).$$

For real values of q and z or, $Fek_{2n}(z, -q)$ is real, so that $Fey_{2n}(z, -q)$ is complex, and will rarely be of importance in the applications. If and when, however, tabulation of functions for imaginary or complex values of z is considered, these relations may be useful.

Functions to be tabulated

Having defined solutions of the several differential equations, we now suggest a set of functions which, in view of technical applications, should be tabulated.

FUNCTIONS

Characteristic Number	Trigonometric	Hyperbolic			
a_{2n}	$ce_{2n}(z, q)$	$Ca_{2n}(z, q)$	$Fey_{2n}(z, q)$	$Co_{2n}(z, -q)$	$Fek_{2n}(z, -q)$
b_{2n+1}	$se_{2n+1}(z, q)$	$Se_{2n+1}(z, q)$	$Gey_{2n+1}(z, q)$	$Co_{2n+1}(z, -q)$	$Fek_{2n+1}(z, -q)$
a_{2n+1}	$ce_{2n+1}(z, q)$	$Ca_{2n+1}(z, q)$	$Fey_{2n+1}(z, q)$	$So_{2n+1}(z, -q)$	$Gek_{2n+1}(z, -q)$
b_{2n+2}	$se_{2n+2}(z, q)$	$Se_{2n+2}(z, q)$	$Gey_{2n+2}(z, q)$	$So_{2n+2}(z, -q)$	$Gek_{2n+2}(z, -q)$

There are some reasons (based upon its physical interpretations) for preferring k to q as a tabular parameter. Apart from the range $0 \leq z \leq \frac{1}{2}\pi$ for the periodic functions, no *natural* upper limits to the ranges of z , $q(0^+ k)$, n exist, and present knowledge of technical applications is too meagre to permit authoritative suggestions as to the useful extent of these ranges. Developments asymptotic in one or more of these variables are known, and tabulation might cease when one or two terms of such a development provide adequate accuracy.

In the first instance, tables of low accuracy, covering wide ranges of z and q , and for the lower values of n , are desirable: these would constitute a numerical exploration of the field, which is essential. It will, however, be justifiable, and in the long run economical, to compute fundamental quantities (especially characteristic numbers) to high accuracy. The ultimate goal is a set of tables interpolable in q (or k) as well as in z , to an accuracy of at least five significant figures, and for sufficient of the lower values of n , say from 0 to 6. The experience of Ince⁸ and HIDAOKA¹² indicates that interpolability in q necessitates an interval of the order 0.1 in this parameter. Consequently, such a set of tables will be laborious to produce, and volumi-

nous to publish. They form no one-man project, so many contributors will be needed. One important purpose of this article is to suggest a suitable scheme into which all contributions could fit, so that both economy of labor and co-ordination of results may be achieved. In any such scheme Ince's Tables³ are foundation-stones, well and truly laid.

Computation: numerical illustrations

Methods for calculating characteristic numbers and the coefficients A, B are well established, and the rate of convergence of the series (12.1)–(12.4) is satisfactory—at least for values of n and q likely to be required.

On the other hand, as has been remarked, the convergence of the various series for the 'hyperbolic' types of function is not everywhere satisfactory. The series involving Bessel function products converge everywhere in the contemplated ranges, and they converge at least as rapidly as the series for the corresponding periodic function. More important still is the fact that they converge with increasing rapidity as z increases, and so behave computationally in a manner similar to asymptotic expansions. We illustrate the behavior of the various developments by giving numerical values of the terms in the series (17.11)–(17.14) for $C_0(z, -4)$ and (19.12), (19.13) for $Fek_0(z, -4)$, for $z = 0$ and for $e^z = 2$ ($z \simeq 0.7$). The values of A_{2r} are taken from Ince's Tables,³ and appear (actually as $(-)^r A_{2r}$) as the terms of (17.11) with $z = 0$.

$C_0(z, -4)$

$z = 0$

r	(17.11)	(17.12)	(17.13)	(17.14)
0	0.55971 72	1.29197 0	0.79002 0	0.83846 0
1	0.59897 00		0.48040 3	0.43686 6
2	0.12051 12		2131 6	1648 9
3	1203 74		23 2	15 6
4	70 68		1	1
5	2 71			
6	7			
	1.29197 04	1.29197 0	1.29197 2	1.29197 2

$C_0(z, -4)$

$e^z = 2$ ($z \simeq 0.7$)

r	(17.11)	(17.12)	(17.13)	(17.14)
0	0.55971 7	+6.30584 0	1.90410 5	2.30875 8
1	1.27281 1	-3.10418 6	1.30948 2	0.95236 4
2	0.96559 6	+ 9060 2	7688 0	3028 6
3	0.38529 .	- 60 2	119 1	25 7
4	9047 .	+ 1 3	7	2
5	139 .			
6	14 .			
	3.289 .	3.29166 7	3.29166 5	3.29166 7

Already, at $z \approx 0.7$, (17.11) is practically useless, and the superiority of (17.14) is becoming evident.

$$Fek(z, -4)$$

$$z = 0$$

$$e' = 2$$

r	(19.12)	(19.13)	(19.12)	(19.13)
0	0.00078 0079	+0.04189 15	0.00025 8013	+0.00227 970
1	130 1687	- 3841 39	39 7057	- 121 818
2	93 6566	+ 607 35	22 9653	+ 8 215
3	67 359.	- 47 80	12 1274	- 230
4	50 25..	+ 2 27	6 306.	
5	38 9...	- 7	3 30..	
			1 7...	
			est. rem. 2 1	
	0.0046.	0.00909 51	0.00114 0...	0.00114 137

At $z = 0$ (19.12) is computationally useless, and even at $e' = 2$ it serves to determine three significant figures only, as against 6 from (19.13). For neither value of z does (19.11) converge.

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¹ W. G. BICKLEY, "The tabulation of Mathieu functions," *MTAC*, v. 1, July 1945, p. 409-419.

² We do, however, replace Ince's θ by q (compare *MTAC*, v. 1, p. 418), thereby avoiding a mixture of English and Greek symbols in the differential equation.

³ E. L. INCE, "Tables of the elliptic-cylinder functions," R. So. Edinburgh, *Proc.*, v. 52, 1932, p. 355-423.

⁴ But these are not the only problems which lead to equations of the group (7)-(10).

⁵ H. E. HEINE, *Handbuch der Kugelfunktionen*, second ed., Berlin, 2 v., 1878-1881.

⁶ *ce* from the initials of 'cosine elliptic,' *se* from those of 'sine elliptic.'

⁷ The superscripts to A or B , denoting the order of the function, will be omitted where this can be done without ambiguity.

⁸ S. GOLDSTEIN, "Mathieu functions," Cambridge Phil. So., *Trans.*, v. 23, 1927, p. 303-336.

* Here, and throughout the article, unless otherwise specified, \sum implies $\sum_{r=0}^{\infty}$.

¹⁰ The letters f , g are convenient and not likely to cause confusion with other mathematical functions.

¹¹ S. GOLDSTEIN, "The second solution of Mathieu's differential equation," Cambridge Phil. So., *Trans.*, v. 24, 1928, p. 223-230.

¹² K. HIDAOKA, "Tables for computing the Mathieu functions of odd order . . .," Imp. Marine Observatory, Kobe, Japan, *Memoirs*, v. 6, 1936, p. 137-157.

Multiplication of Matrices

A method for finding rapidly the product of two matrices has been developed at the Ballistic Research Laboratory, Aberdeen Proving Ground. The method uses punched cards and a new machine built by the IBM Corporation for the Laboratory.

Each element of each matrix is punched on a separate card together with its row and column number and other identifying material. The cards representing matrix A are sorted by rows and placed in one of the two feed hoppers of the machine, the cards of matrix B are sorted by columns and placed in the other feed hopper. The machine, if properly wired, will feed cards from both hoppers, take the product of the two numbers which it reads out of the two cards (at the same time comparing the column index of the A card with the row index of the B card, and stopping if by some mistake these two indices are not equal), and accumulate the products as long as these indices increase. After the first n cards (i.e. when the compared indices drop from n to 1) it will store the number accumulated up to this time (which is c_{11} , the first element of the product matrix $C = AB$), start accumulating anew and punch c_{11} on the next card of the B stack into an unused part of the card. In the same way the machine computes and punches c_{21}, \dots, c_{n1} .

Before beginning the operation, the last n cards (representing the n th column) of B are duplicated and placed at the end of the stack. After n^2 cards have passed through the machine, or at any earlier time, the cards which have passed through are taken out of their stackers and put back into the feed hoppers. Because of the extra n cards in B , the two stacks are now out of phase by n , and the machine, without any change in wiring or without even stopping, computes $c_{1,n}, c_{2,n}, c_{3,n}, \dots, c_{n,n-1}$. It punches each of these numbers "at the first opportunity," i.e. into the first card of B which does not yet have a c_{ij} punched into it. The process is repeated until all elements of C are computed and punched. This is done when the $n^2 + n$ cards of B have passed through the machine n times. (If the machine is left running at that time, it will continue to compute but will not punch any more results.) The $n^2 + n$ cards of B now contain all elements of C , the elements of the n th column appearing twice. Thus the cards can be used for another multiplication in which C is the right-hand factor. The wiring is such that whenever the machine punches a number c_{ij} , it also punches the indices i and j .

The time required for the operation varies approximately as the cube of n . For multiplying up to six significant figures the machine feeds 6,000 cards per hour; for more than six significant figures but not exceeding twelve, the rate is 3,000 per hour. Thus two 9-row matrices with elements of not more than six significant figures can be multiplied in about nine minutes (the significant figures of all elements should be in the same six decimal positions). Because of the necessity of sorting the cards, of checking and correcting errors, etc., the actual time required for the whole operation may well be several times as long as the calculated running time of the machine. This is partly due to the fact that the machine is still in its experimental stage. Even so, this method is believed to be considerably faster than any other method in use at this time.

It is possible to set the machine in such a way that it will produce $kI - C$ instead of C . This is useful in finding the inverse X of a matrix A by a process of successive approximations such as $X_{n+1} = X_n(2I - AX_n)$. (See H. HOTELLING, "Some new methods in matrix calculation," *Annals Math. Statistics*, v. 14, 1943, p. 1f.)

The machine is not specifically designed for the purpose of multiplying matrices. It is a general purpose computer, capable of handling such problems as projective transformations, third order interpolations and others.

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RECENT MATHEMATICAL TABLES

233[A-E, G-I, K-N, U].—A. FLETCHER, J. C. P. MILLER, & L. ROSENHEAD, *An Index of Mathematical Tables*, London, Scientific Computing Service, 23 Bedford Square, W.C. 1, 1946. viii, 450 p. 15.6 × 25.3 cm. 75 shillings.

Five years ago there were less than ten experts deeply versed in the international field of mathematical tables and their reliability. Of these experts, often responding to requests for information, one was L. J. COMRIE, the publisher of the work under review. With war's advent research workers on its problems were on every hand clamoring for enlightenment as to the best tables, if any, of this or that function, of such and such a range of argument. At least four countries soon felt that something ought to be done to assemble and to disseminate tabular information.

The National Research Council of the United States was the first to effect some relief in this regard, by the foundation three years ago of the quarterly publication, in which this review appears, and which has recently completed its first volume of nearly 500 pages. England came next, in 1943, with first printers' proofs of the *Index* now before us, a work by three well-known members of the department of Applied Mathematics at the University of Liverpool. Assembly of material which later developed into the *Index* was begun some years before, and the publisher expected that the volume would be ready for distribution early in 1944, but insurmountable difficulties besetting English printers delayed publication for two years. On account of this delay, proofs were revised so as to take account of new material to the end of 1944. Thus we now have the first work of its kind. During the past year and a half the reviewer has become thoroughly familiar with this publication through checking of early proof and, practically the final proof, which is now before him. In our limited space it will be quite impossible adequately to suggest the extraordinary richness of this epochmaking work, filled with so many excellencies for guiding to his goal the searcher for existing printed or ms. tabular aid.

The work contains the following two main divisions: Part I, p. 18-372, Index according to Functions; and Part II, p. 373-444, Bibliography. We shall first consider the latter. Here is a complete list of the published material referred to in Part I, arranged alphabetically according to authors. In the case of books or pamphlets brief titles are given, but in the case of tables, or other articles, in serials, the titles of the articles are rarely given, but only other bibliographical details. Under each name the entries are arranged chronologically, so that the first entry is always the year of publication (a few exceptions which we shall not explain), then the name of the serial, the series, the volume number, and inclusive pages of the article containing the table or other material. When several entries were published in the same year, they are differentiated by the different entries; for example, under Airey we have five entries 1911a, 1911b, 1911c, 1911d, 1911e; and also 1918, 1918a showing that with more than one entry for a given year the first entry is not necessarily associated with some letter.

In the case of books or pamphlets the name of the publisher is given, in the case of books published comparatively recently, and when there is more than one edition, the date of the first edition is stated; but often a very different date, that of a later edition actually consulted, may be the year in the reference.

For one work an exception is made to the general rule of referring to only a single edition. The need for such an exception in the case of Jahnke & Emde's work (where all material is indexed) is obvious; thus the dates of the first three editions are given, namely 1909, 1933, 1938.

The number of references (68) under "GLAISHER, J. W. L." are more numerous than those under any other name. Each entry does not necessarily refer to a table. For example, 1873 is Glaisher's famous report on tables for the BAASMTTC; 1873a is an article "On the progress of accuracy of logarithmic tables," 1873b is an article "On logarithmic tables," 1882 and 1888 are the articles on "Logarithms," and "Tables, Mathematical," in the *Enc. Brit.*, ninth ed.; 1911a and 1911b are similar articles in the *Enc. Brit.*, eleventh ed. And so with many other entries throughout the Bibliography; the only reference under Horsburgh is 1914, the *Napier Tercentenary Exhibition Handbook*; under Mehmke, R.—d'Ocagne, M. 1909, "Calculs Numériques" in *Encycl. d. Sci. Math.*; and under Henderson, J. 1926, *Bibliotheca Tabularum* . . . and 1929, "Mathematical Tables" in *Enc. Brit.*, fourteenth ed. There are many references to tabular errata.

F. M. & R. are careful to make it perfectly clear, by means of a prefixed asterisk, when any item has not been seen, so that any information given is indicated as second-hand. There are only slightly over 300 asterisks in more than 2000 entries. Some reference entries are to mss. only, for example, under Fischer, Kohler, Neville, Spenceley, Thomson, J., and Tweritin; but the Bibliography indicates only a fraction of the mss. referred to in Part I. For example, although dozens of ms. tables by J. C. P. Miller are there listed, all we find under Miller's name in Part II is "Several unpublished tables have also been included." And so also in the case of other authors.

Part II contains various special statistical tables not indexed in Part I. It also includes a considerable number of books on probability and statistics in the wider sense, ranging "from treatises on fundamentals down to elementary works concerned mainly with applications to education, psychology, medicine and other subjects; some of the latter owe their inclusion to an accurate table of the error integral. Books on probability of an entirely non-mathematical kind have normally been excluded." Books on nomography and graphical methods are not generally included in Part II, nor references to tables in the Theory of Numbers, so completely dealt with by D. H. LEHMER, 1941.

In the Bibliography are some comprehensive headings. For example, under "British Association Mathematical Tables," are listed the BAASMTTC volumes indicated by B.A. 1, 1931, B.A. 2, 1932 . . . B.A. 9, 1940. Then there are the inclusive page numbers for the BAASMTTC reports in B.A.A.S. Reports 1873, 1-175; 1875, 305-336; . . . 1939, 327-329. Under "New York W.P.A." are listed various volumes of the NYMTP, and under "Tracts for Computers" references are given to the authors of the 24 Tracts from FAIRMAN, 1919, to KENDALL & SMITH, 1939.

The authors themselves confess that it is difficult to define the scope of their Bibliography. Old tables are included only when they seemed to be still of interest, such as the outstanding Briggs tables of trigonometric functions, 1633. Thus many of the tables described in detail by Glaisher's 1873 report are not considered. Minor tables, published during the last few decades, however, have not been "lightly rejected." The authors tell us that altogether they intentionally rejected about 1000 tables; but the reviewer has found little cause for criticism on the basis of such omissions.

In alphabetical arrangement, very frequently ã is not regarded as æ; or ð and ø [ð] is substituted for this in the *Index* as oe; or ù as ue.

It is, of course, not surprising that TÖLKE's *Praktische Funktionenlehre*, v. 1, 1941 (see RMT 240), is not listed; nor J. PETERS, *Siebenstellige Werte der trigonometrischen Funktionen in Tausendstel zu Tausendstel des Neugrades*, Berlin, Landesaufnahme, 1941; nor I. M.

RYZHIK, *Tablitsy*, see RMT 219. But one might have expected a record of the New York edition, 1944, of Lindman's "Examen des *Nouvelles Tables d'Intégrales définies* de M. Bierens de Haan," and certainly of F. W. Newman's *Mathematical Tracts*, 2 v. Cambridge, England, 1888-1889; see *MTAC*, v. 1, p. 456 where several of its tables are listed. Further comment on the Bibliography is reserved till later in the review.

Let us now consider Part I. This is divided into 24 Sections, with the following general titles:

- 1 Primes, Factors, Products and Quotients
- 2 Powers, Positive, Negative and Fractional
- 3 Factorials, Binomial Coefficients, Partitions, etc.
- 4 Bernoulli and Euler Numbers and Polynomials. Sums of Powers and of Inverse Powers. Differences and Derivatives of Zero, etc.
- 5 Mathematical Constants, π , e , M , γ , etc.; Multiples and Powers. Roots of Algebraic and Transcendental Equations. Miscellaneous Constants. Conversion Tables
- 6 Common Logarithms, Antilogarithms, Addition and Subtraction Logarithms
- 7 Natural Trigonometrical Functions. Miscellaneous Functions connected with the Circle and the Sphere
- 8 Logarithms of Trigonometrical Functions
- 9 Inverse Circular Functions. Trigonometrical Functions of Two or Three Arguments (including Products, Solutions of Plane and Spherical Triangles, etc.). Sexagesimal Interpolation Tables
- 10 Natural and Logarithmic Values of Exponential and Hyperbolic Functions
- 11 Natural Logarithms of Numbers and of Trigonometrical Functions. Inverse Hyperbolic Functions
- 12 The Gudermannian, Combinations of Circular and Hyperbolic Functions, Circular and Hyperbolic Functions of a Complex Variable
- 13 Exponential and Logarithmic Integrals, Sine, and Cosine Integrals, etc.
- 14 Factorial or Gamma Function, Psi Function, Polygamma Functions, Beta Function, Incomplete Gamma and Beta Functions
- 15 The Error Integral, Higher Integrals, Derivatives, Hermite Polynomials and Functions. Moments
- 16 Legendre Functions
- 17 Bessel Functions of Real Argument
- 18 Bessel Functions of Pure Imaginary Argument, or Modified Bessel Functions
- 19 Bessel Functions of Complex Argument. Kelvin Functions
- 20 Miscellaneous Bessel and Related Functions
- 21 Elliptic Integrals, Elliptic Functions, Theta Functions
- 22 Miscellaneous Higher Mathematical Functions
- 23 Interpolation, Numerical Differentiation and Integration, Curve-Fitting
- 24 Tables and Schedules for Harmonic Analysis, etc.

Part I is the paramount feature of the Index. What we now immediately set forth will be largely extracted from part of the clearly articulated Introduction (p. 1-15). Each section is made up of a number of articles, and each article deals with tables of one function or of a strictly limited group of functions. The articles are numbered in a decimal notation, the integral part being the number of the section. In any section the articles which have the same digit in the first decimal place may be regarded as forming a subsection, and a list of such subsections is usually given in the introduction to the section, as 10.0 and 23.0, of Sections 10 and 23 respectively.

Most of the articles consist essentially of lists of tables which give the same function (or sometimes very closely related functions). The order of the items is normally that of the number of decimals or figures given. Each item generally occupies one line, and gives, in order from left to right, information about (1) number of decimals or figures; (2) interval and range of argument; (3) facilities for interpolation; (4) authorship and date in the form used in Part II.

When values such as $0^{\circ}.01$, 1° , $0^{\circ}.1$, $0^{\circ}.01$, etc. stand in the first column the table in question gives values to hundredths of a degree, to grades, to minutes of arc, to tenths of a second of arc, to tenths of a minute of time, to hundredths of a second of time, etc.

The symbols in the column (usually immediately preceding the name of the author) give information about means of interpolation. No less than 40 symbols employed in this connection are listed (p. 16-17) and suggest the painstaking record prepared to inform the computer in advance of procuring the table. A few of these symbols may be selected at random by way of illustration, namely:

Δ^1 First, second and third differences. Similarly for higher indices.

Δ^* Differences as far as needed; usually to different orders in different parts of the range.

Δ^2 Second differences only.

Δ^4 Second and fourth differences.

Δ^{**} Even differences, as far as needed, etc.

PP Proportional parts giving every tenth of some differences, rounded to the nearest integer.

fPP Full proportional parts, all tenths of all differences (except perhaps very small ones) occurring in the table, rounded to nearest integer.

The whole of Section 23 is devoted to Tables and Notes on Interpolation, Numerical Differentiation and Integration, Curve-Fitting.

The different sections naturally vary considerably in length. After 22 pages devoted to Sections 1-3 we come to the excellent Section 4, the longest of all (41 p.), where a great amount of material is assembled, with varying notations explained, and explanatory text scattered throughout. The main subdivisions of the section are as follows:

4.0 Introduction, Definitions and Interconnections

4.1 Bernoulli Numbers and Rational Multiples

4.2 Euler Numbers and Rational Multiples

4.3 Glaisher's I , T , P and Q Numbers and Rational Multiples. Other Special Numbers

4.4 Bernoulli Polynomials; Sums of Integral Powers of Integers

4.5 Euler Polynomials

4.6 Sums of Inverse Powers of Integers, etc.

4.7 Sums of Products involving Primes only

4.8 Irrational Multiples of Bernoulli Numbers, etc., or of Sums of Inverse Powers of Integers

4.9 Generalized Bernoulli and Euler Numbers and Polynomials. Differences and Derivatives of Zero. Coefficients in Formulae involving Derivatives and Integrals in Terms of Differences.

Subheadings under 4.02, p. 40-45, for example, are

4.021 Bernoulli, Euler and Other Numbers

4.022 Sums of Inverse Powers

4.023 Generalized Bernoulli and Euler Numbers. Differences of Zero, etc.

In 4.15 are tables of Tangent Numbers; and in 4.171, of Genocchi's Numbers $= (-1)^n G_{2n}$ (Lehmer) $= \lambda_n$ (Sheppard) $= 2(2^{2n} - 1)B_{2n}$, the simplest integer set of multiples of Bernoulli numbers.

This Section, as well as many another in the volume, may become a valuable reference source in the field under review. The care with which varying notations are set forth in the *Index* may be further illustrated by the five symbols for $d(\ln x!)/dx$ in Section 14, and by turning to Section 16 on Legendre Functions. Thus in many tables studied where notations are unfamiliar our friends F. M. & R. are likely to be able to render much more than "first aid."

Next, let us consider the headings in Section 7 with its admirable 7.0 Introduction. Then 7.1 Tables with Argument in Radians [7.11 $\sin x$ and $\cos x$; 7.12 $\tan x$; 7.13 $\cot x$;

7.14 Sec x ; 7.15 Cosec x . 7.2 Tables with Argument in Degrees and Decimals [7.21 Sin x ; 7.22 Tan x ; 7.23 Sec x]. 7.3 Tables with Centesimal Argument [7.31 Sin x ; 7.32 Tan x ; 7.33 Sec x]. 7.4 Tables with Argument in Degrees, Minutes, etc. [7.41 Sin x ; 7.42 Tan x ; 7.43 Sec x]. 7.5 Tables with Argument in Time. 7.6 Auxiliary Functions (mainly for Small Angles), All Arguments [7.61 Functions σ and τ ; 7.62 cosec $x - 1/x$ and $1/x - \cot x$; 7.63 $\sin x/x$, also $\cos x/x$ and $\sin^2 x/x^2$; 7.64 $\tan x/x$; 7.66 $x - \sin x$; 7.67 $\tan x - x$; 7.68 Berry Functions]. 7.7 Versed sines, Haversines, etc. All Arguments [7.71 Vers x ; 7.715 $2 \sin^2 \frac{1}{2} x$ Cosec $1''$; 7.716 $2 \sin^4 \frac{1}{2} x$ Cosec $1''$; 7.718 $\tan^2 \frac{1}{2} x$ Cosec $1''$; 7.72 Hav x]. 7.8 Miscellaneous Tables connected with the Circle and the Sphere (Areas, Volumes, Chords, Segments, Regular Polygons, etc.). [7.811-7.824 Quantities associated with the whole Circle or Sphere; 7.83 Arcs, Chords, etc. of Circles; 7.84 Areas of Segments of a Circle; 7.85 Regular Polygons; 7.86 Surface of Zone of Sphere; 7.87 Volume of Segment of a Sphere]. 7.9 Miscellaneous Expressions involving Circular Functions.

Notes may here be made concerning several things of this Section which are characteristic of most others. Names of authors and dates printed in bold type in various articles, indicate that the corresponding tables are outstanding of their kind. This emphasis has been used particularly in respect to tables of elementary functions such as logarithms and trigonometric functions. Again, there are many places where a number follows the year of publication, such as, p. 129, Norie 1938 (369); this refers to p. 369 of Norie's book of 1938. Another important element of the volume, illustrated in this Section, consists in the notes with regard to errors of tables listed; e.g. in 7.21 and 7.22 after tables of Briggs 1633, explicit errors are noted, and in 7.41 after Hudson & Mills 1941, is "On errors see *M.T.A.C.*, 1, 86-87, 1943." Not infrequently in the *Index* the range recorded is smaller than that of the original table. This indicates that the editors know that other decimal-places, or for other parameters, values are incorrect.

Now, turning to other sections, again and again the reviewer has been exasperated by the practical impossibility of finding out what has been tabulated in tables edited by Karl Pearson. Now he may, with a sigh of relief, turn to the *Index*, and find out!

Not a few users of the *Index* are likely to take it up to discover the definitions of terms, such as, for example: DuBois-Reymond constants (p. 89), Lambertian (p. 183), Ser's Integral (p. 196), Havelock's Integral (p. 302), Hogner's Integral (p. 307), White-Dwarf Functions (p. 333), Fermi-Dirac Functions (p. 340), Clausen's Integral (p. 344). An excellent *Index* to Part I (p. 445-450) gives one an immediate reference to the article for any particular topic.

Perhaps it is worth while to jot down a few miscellaneous comments.

P. 25, 99-100. To tables for solving cubic equations those by F. KERZ, 1864, A. HAMILTON, 1899, and H. A. NOGRADY, 1936, might well have been added; see RMT 249, and 218.

P. 33. The attribution of tables of $n^{1/3}$ and $n^{2/3}$ to IVES, *Mathematical Tables*, 1934, is misleading, since the tables were by M. MERRIMAN; see RMT 235.

P. 65. Under $\sum_n = 1/2^n + 1/3^n + 1/5^n + 1/7^n + \dots$, the first two tables listed are Euler 1748 (237), $n = [2(2)36; 15D]$, and Glaisher 1877f (175) $n = [2(2)36; 15D]$. Glaisher reprints Euler's table with 14 corrections; hence F. M. & R. reasoned that a recomputation should be under Glaisher's name only. There are similar cases of this kind scattered throughout the volume.

P. 104. Zeros of $\sin z \pm z$ by J. FADLE, *Ingenieur-Archiv*, v. 11, 1940, p. 129 (see N 50) have been overlooked.

P. 105. The heading of article 5.745 is $10^x \log_{10} x = x$ or $10^{x^*} = x^{10}$.

P. 173-174. One wonders why the headings of paragraphs 10.51, 10.52, 10.54, 10.55 are not, as everywhere else (seemingly), more specific in indicating that the logarithms are to the base 10.

P. 308. While Müller's tables of $y = \frac{1}{2}\pi\{I_0(x) - L_0(x)\}$ and y' are listed, those of Zurmühl (see N 48) are not.

P. 380. Surely the entry under Briggs & Gellibrand 1633 should have been more properly under Briggs, with a statement that G. contributed part 2 of the text. Several misleading references to tables by Briggs & Gellibrand would then have been avoided.

P. 392. Under Euler 1748 why not add *Opera Omnia*, (1), 8, 1922, Leipzig, Teubner as is done similarly under 1755? Moreover, exactly this reference is given on p. 65.

P. 401, 409, 412. If under Lommel there is a reference to *Munich Abhandlungen*, why under Kulik 1866 is it not *Vienna Sitzungsber.* instead of *Wiener [not Wien] Sitzungsber.*, as under Herrmann?

P. 404. Under "Hutton, C" is the entry "1775 Table of reciprocals and square roots in Hutton's *Miscellanea Mathematica*, 4"; "4" has no meaning in this connection. *M.M.* was a periodical issued, 1771-1775, in 13 numbers, containing 55 "articles." Since "A table of square roots and reciprocals of all numbers from 1 to 1000" is in article 55, p. 329-342, it was printed in no. 13, 1775. For fuller details regarding *Miscellanea Mathematica* see our notes in *Math. Gazette*, Apr. 1929, p. 388.

P. 409. Kulik's table in the Prague *Abhandlungen* occupies p. 21-123 (not 1-123); see RMT 249.

P. 443. The entry under Wronski, H. should be transferred to p. 402 under Hoëne-Wronski, J. M., since Hoëne was the family name, later expanded by the author in question. Further, in referring to S. Dickstein's Polish edition of Hoëne-Wronski's table of logarithms, it would be better to add "abridged," since iv + 64 p. + tables on 6 plates, are abridged to xv p. + 1 plate.

P. 443. Under "Yano, T., *Calculating Tables*," the text is misleading in more than one respect. First of all, the translation of the title is "Multiplication, Division and other Tables, with an Appendix of logarithmic tables, conversion tables of measures, conversion tables of monetary units," "compiled" by TSUNETA YANO, Tokyo, Hakubunken & Co., 1913. ii, 1817 p. On the back binding of the book is the following: *Tables de Calcul. Rechentafeln.* [And in Japanese, the title-page title]. Pages 2-1799 are identical, in display and otherwise, with the tabular pages of the two volumes of the original 1820 edition of A. L. Crelle's *Rechentafeln*, except that the printer's errors which Crelle listed have been corrected, and the Japanese edition had (in Japanese) "end figures" printed (where Crelle had a blank space) at the head of the two-figure column on the right-hand side of each page. Hence, since all but the Appendix, p. 1806-1815, was by Crelle, and taken over by Yano in his *compilation*, there should certainly be an indication of this fact and a reference to the work under Crelle's name.

Most of these comments on lacunae and minor matters are with reference to publications which the authors had not seen, and naturally do not in any way detract from the real value of this truly remarkable work. Although we have checked hundreds of the references in Part I, we do not now find any errors worth noting. The extraordinary accuracy and comprehensiveness of the work, compiled during five or six years, when the authors were also making their contributions to the war effort, must excite profound amazement on the part of anyone who has attempted something of a similar nature. Everyone must also be deeply impressed by the notable typographical displays, greatly facilitating ready use, and tending to a marked degree to clarity of presentation of material often complicated.

Since practically every table listed in the *Index* is in the Brown University Library, and since microfilms or photoprints of any table (not copyrighted) may be purchased by anyone from the Photographic Laboratory, Brown University, Providence, R. I., one having access to a copy of the *Index* may readily find what table he needs, and procure a copy of practically anything listed. Thus the *Index* is an indispensable tool for every university library, and for every research center where computing is carried on extensively. Many an individual will probably feel that he also must have the volume constantly on his desk, even though the initial outlay be considerable; instruction received from the volume, and time saved by its use would soon yield large dividends on the investment. [See also p. 64.]

R. C. A.

234[A, D].—M. E. LONG, *Tables of the Gamma Function for Complex Argument*. Offset print of typescript on one side of each of the 10 sheets + Amendment no. 1 (errata). Radar Research Development Establishment, Memorandum no. 96, The Malvern, Worcestershire, England, May, 1945. 21.5×34.3 cm. These tables are available only to certain Government agencies and activities.

There are tables, to 4D or 5D, of $\ln |\Gamma(z)|$ and $-\arg \Gamma(z)$, $z = x + iy$, against $x = -5(1) + 5$, $y = 0(1)1$; additional entries have been made for $x = .25$ and $.75$.

The entries have been checked by differencing, and further checks were made for special values of the argument by use of the following relations:

$$\begin{aligned} |\Gamma(1 + iy)| &= y|\Gamma(iy)|, \arg \Gamma(1 + iy) = \frac{1}{2}\pi + \arg \Gamma(iy), \\ |\Gamma(1 + iy)| &= [xy/\sinh \pi y]^{\frac{1}{2}}, |\Gamma(\frac{1}{2} + iy)| = (\pi/\cosh \pi y)^{\frac{1}{2}}, \\ |\Gamma(\frac{1}{2} + \frac{1}{2}iy)| |\Gamma(1 + \frac{1}{2}iy)| &= \pi^{\frac{1}{2}} |\Gamma(1 + iy)|. \end{aligned}$$

The value of $\Gamma(-x)$ was obtained from H. T. Davis, *Tables of the Higher Mathematical Functions*, v. 1, 1933, also the values for $\Gamma(iy)$ were checked against those given in that book. Compare RMT 195.

It is believed that the Tables are subject to errors of not more than half a unit in the fourth decimal place and of not more than two units in the fifth where this has been quoted.

Extracts from introductory text.

235[B].—GREAT BRITAIN, Farnborough, Royal Aircraft Establishment, *Table of Two-Fifths Powers, for Use in Compressibility Calculations*. Offset print of typescript on one side of each of 7 sheets. Report no. S.M.E. 3329, May, 1945. 21.5×34.3 cm. These tables are available only to certain Government agencies and activities.

The table covers the range normally needed in compressibility work, $x^{2/5}$ for $x = [.5(.005)1(.01)2; 7D]$, Δ . The error is everywhere less than one unit in the seventh decimal place and, in general, is less than one half a unit. The table was prepared from key values computed to ten significant figures. These were then subtabulated to tenths, from fourth differences, on a National Accounting Machine.

Notes on the table

1. The following multipliers may be used to extend the range:

x	$x^{2/5}$	x	$x^{2/5}$
.1	.398 107 1705	10	2.511 886 432
.25	.574 349 1774	25	3.623 898 318
2.5	1.442 699 9056	100	6.309 573 446

2. Single values can be found to any needed accuracy by repeating the correction formula $u = A[x/y^{1/5} - y]$, where y is the estimate to $x^{2/5}$, and u is the required correction. If y is correct to n digits, $y + u$ will be correct to $2n - 2$ digits at least.

Extracts from introductory text.

EDITORIAL NOTE: A table of $x^{2/5}$ by M. MERRIMAN was published on p. 26 in the first and later editions of his *American Civil Engineers' Pocket Book*, New York, 1911, 1912, 1916, 1920, 1930. This table, computed specially for the *Pocket Book*, called *Handbook*, in its latest edition, was for $x = [.1(.1)8(.2)10; 4D]$. Merriman remarked that to find $(10n)^{2/5}$ multiply the tabular number by 2.5119; to find $(0.1n)^{2/5}$ multiply by .39811. This table was copied (with permission) by H. C. IVES in his *Mathematical Tables reprinted from Searles and Ives' Field Engineering with Additions*, New York, 1924, p. 113; and second ed., 1934. The table was not in the work of SEARLES & IVES.

There is also a table of x^* by F. EMDE in E. JAHNKE & F. EMDE, *Tables of Functions*, 1933, 1943, 1945 (Addenda, p. 8), for $x = [.1, .5(1)1(.2)3(.5)5, 10; 4D]$.

236[B, P].—JOSEPH H. KEENAN & JOSEPH KAYE, *Thermodynamic Properties of Air including Polytropic Functions*, New York, Wiley, and London, Chapman & Hall, 1945. iv, 73 p. 19 × 25.4 cm. \$2.25.

This volume, for the mechanical engineer specialist, is by professors at the Massachusetts Institute of Technology. Extracts from the Preface: "The need for a working table of the thermodynamic properties of air has been emphasized recently by the rapidly growing interest in the gas turbine. The compression of atmospheric air which occurs in a gas turbine can be computed with reasonable convenience without the aid of a table. Computations of certain other processes, however—such as the heating of air in a regenerator or the expansion of air and similar gases from states of high temperature—involve laborious integrations if tabulated properties are not available.

"For such computations air may be considered to be a relatively simple substance because the expression $pv = RT$ is an adequate equation of state. By virtue of this fact the table of properties for air is simpler than the corresponding tables for vapors in that a single independent argument serves in place of two. Within the same space, therefore, a far more detailed table is possible for air than for steam. In the present table, as contrasted with existing tables for vapors, interpolation can often be dispensed with, and where it must be used it is a single interpolation which may be done by inspection."

Table I, "Air at Low Pressures," p. 3-33, was previously published in much abbreviated form in *J. Appl. Mech.*, v. 10, 1943, p. A123-A130. The present form involved revisions, entire recomputation, interpolation to smaller intervals, and extension to higher temperatures. In the Table, for T (temperature, deg. F abs) = 300(13000)(10)6500, values are given for t (temperature, deg. F), $1D$; h (enthalpy per unit mass, Btu/lb), $2D$; p_r (relative pressure, 4-5S); u (internal energy per unit mass, Btu/lb), 4-6S; v_r (relative volume) 4-5S; $\phi = \int_{T_0}^T c_p dT/T$, Btu/lb F abs (c_p = specific heat at constant pressure), 5D.

Table 2, " $R \ln N/778$," $N = [1(.001)1.2(.01)10; 5D]$, where R (gas constant for air) ~ 53.3 .

Table 3, Miscellaneous.

Table 4, For $n = [1.05(.05)1.8; 4-5S]$, n^{-1} , $(n-1)^{-1}$, $1/p$, $[2/(n+1)]^p$, $R/(n-1)$, R/p , $[2gR/p]^{\frac{1}{2}}$, $[2g/(pR)]^{\frac{1}{2}}$, $g = 32.2$, $p = (n-1)/n$.

Table 5, For $r = [0(.0001)0.004(.0002)0.01(.0005)0.04(.001)0.07(.002)2(.005)9(.002)96(.001)1; 3-5S]$ are given the values of r^{-1} , $r^{1/2}$, $r^{1/3}$, $(1-r^{1/2})^{\frac{1}{2}}$, $r^{1/3}(1-r^{1/3})^{\frac{1}{3}}$.

Tables 6-9 (p. 42-45). Values of $r^{1/n}$, r^p , $(1-r^p)^{\frac{1}{p}}$, $r^{1/n}(1-r^p)^{\frac{1}{p}}$, $p = (n-1)/n$, for $r = [0(.02)1; 4D]$, $n = 1.05(.05)1.8$; also r^{-1} to 3S.

Table 10 (p. 46-49). "Log N ," for $N = [1(.001)2; 4D]$ and $N = [1(.01)9.99; 4D]$ from E. V. HUNTINGTON, *Four Place Tables of Logarithms and Trigonometric Functions*, Boston, 1910.

Table 11 (p. 50-51). " $L_N N$," $N = [1(.01)10; 4D]$, from L. S. MARKS & H. N. DAVIS, *Tables and Diagrams of the Thermal Properties of Saturated and Superheated Steam*, London, 1909.

Table 12, "Conversion Factors," and **Table 13**, "Temperature Conversion." "Sources and Methods" and "Examples," p. 56-71. "Acknowledgments" and "Bibliography," p. 72-73.

The authors of this volume call the functions of r in Table 5, "polytropic functions"; they are based upon the equation $pv^n = \text{const}$.

R. C. A.

237[D].—CARL R. ENGLUND, "Dielectric constants and power factors at centimeter wave-lengths," *Bell System Technical J.*, v. 23, 1944, table on a plate 28 × 22 cm. bet. p. 118 and 119.

This table is of $\pi \tan \pi$, $\pi = [0(.001).519; 8D]$.

238[D].—NYMTP. *Table of Arc Sin x*, New York, Columbia University Press, 1945, xix + 121 p., 17.8×26.2 cm. \$3.50.

The main table of this volume gives to 12D the principal (radian measure) values of the function

$$\sin^{-1} x = \int_0^x (1-t^2)^{-1/2} dt = \frac{1}{2}\pi - \cos^{-1} x,$$

together with values of its second central difference δ^2 . The argument interval is .0001 except near $x = 1$ where interpolation becomes difficult and the interval is made ten times finer. More precisely, the table is for

$$x = [0(.0001).9890(.00001)1].$$

Auxiliary tables are given as follows:

(1) The function $f(v) = 1 + v/12 + 3v^2/160 + 5v^3/896 + \dots$ for $v = [0(.00001).0005]$ to 13D. This is intended for use when x is extremely close to 1 by means of the formula

$$\sin^{-1}(1-v) = \frac{1}{2}\pi - (2v)^{1/2}f(v).$$

(2) Tables of interpolation coefficients for use in Everett's formulae involving second and fourth order differences (actually 50 values of δ^4 appear on the last page of the main table).

(3) Conversion tables: degrees, minutes, and seconds into radians and vice versa, together with the first 100 multiples of $\frac{1}{2}\pi$, for use in pencil and paper work.

Many future users of the main table will not require the full 12D accuracy offered. Simple linear interpolation is good to 8D, or more, for $0 < x < .81$ and (on account of the finer argument interval beyond .989) also for $.989 < x < .991$. Seven decimals are assured for $0 < x < .958$ and for $.989 < x < .998$. Values obtained by linear interpolation always err in excess.

This table is a companion volume to the recent NYMTP table¹ of $\tan^{-1} x$.

The occurrence of the function $\sin^{-1} x$ is, of course, wide-spread especially among problems dealing with portions of circles (e.g. area of a segment of a circle, moment of inertia of an ogive). It also occurs in the evaluation of a large class of common indefinite integrals. Hence this table, which gives *directly* this important function, should be of considerable use to the practical computer.

There is a bibliography of 11 tables, and charts having to do with $\sin^{-1} x$. The most extensive of this is due to HAYASHI² and is a 7D table with interval .001 (except near $x = 0$). There is also the recent table of R. A. DAVIS³ with the same interval but only to 6D. Hence the present table supersedes all previous ones by a wide margin. There is also every reason to believe, in view of the method of computation, that the table is accurate to within .51 units in the last decimal place. We have come to expect both these features, accuracy and extensiveness, from the 17 major volumes already put out by NYMTP. The unexpected feature of this volume is in the color of its binding: instead of the traditional buff buckram, so familiar by now, it is bound in a not unattractive shade of red.

D. H. L.

¹ New York, 1942; see *MTAC*, v. 1, p. 47-48.

² K. HAYASHI, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion*, Berlin, Springer, 1926, p. 5-7, 12-50, 52-86.

³ R. A. Davis, *Table of Natural Sines and Radians*, Oakland, Cal., Marchant Calculating Machine Co., 1941, 8 p., no. MM193; see *MTAC*, v. 1, p. 94f, 124.

EDITORIAL NOTE: The following tables of $\sin^{-1} x$ with radian argument are not mentioned in the NYMTP Bibliography: (a) H. B. DWIGHT, *Mathematical Tables*, New York and London, 1941, p. 114, $x = [0(.001)1; 4D]$; (b) F. EMDE, *Tables of Elementary Functions*, Leipzig and Berlin, 1940; Amer. ed., 1945, p. 98-99 (see *MTAC*, v. 1, p. 384-385), $x = [0(.01).8; 4-5D]$, also graph; (c) G. PRÉVOST, *Tables de Fonctions Sphériques*, Bordeaux and Paris, 1933, p. 134*, $x = [0(.01)1; 5D]$.

The earliest table of $\sin^{-1} x$, with radian argument, appears to have been that of K. HAYASHI, in his *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen* . . . , Berlin, 1926. The first table of $\sin^{-1} x$ seems to have been that of ABU'L HASAN of Morocco, in the 13th century, with argument $0(15')59''15'$; see A. v. BRAUNMÜHL, *Vorlesungen ü. Geschichte d. Trigonometrie*, Berlin, v. 1, 1900, p. 84. This table occupies v. 1, p. 121-124 of J. J. Sédillot's French translation from the Arabic of Hasan's *Traité des Instruments Astronomiques*, Paris, 1834.

- 239[D].—(1) WALTER JAMES SEELEY (1894—), *Impedance Computing Tables*, Engineers' Publishing Co., 401 North Broad St., Philadelphia, Pa., 1936, 24 p., 14.1 × 21.6 cm. 25 cents. (2) J. C. P. MILLER, *Tables for Converting Rectangular to Polar Coordinates*, Scientific Computing Service, 23 Bedford Square, London, W.C. 1, 1939, 15.3 × 25 cm. Now 2 shillings and 6 pence. Authorized American Reprint, Dover Publications, 1780 Broadway, New York 19, [1943]. (See also RMT 141, *MTAC*, v. 1, p. 177–178.)

The object of each of these tables is to facilitate the conversion of pairs of quantities (a , b) to the polar form (z , θ) where

$$z = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}(b/a).$$

Tabulation of the angle θ presents no difficulty, but the modulus z needs a table of double entry for full, direct tabulation, and this requires a prohibitive amount of space. In a letter to *Electrical Engineering* for August, 1933, v. 52, p. 583–584, W. J. Seeley gives a summary of methods whereby the use of a double-entry table may be avoided; subsequent issues (v. 52, p. 724, 799–800, 936; v. 53, p. 228) contain contributions by others interested in the topic. Seeley's preference is given to the following method:

Write l for the larger and s for the smaller of $|a|$ and $|b|$ (Seeley considers, in fact, only the case $a \geq b \geq 0$). Then

$$(I) \quad z = \sqrt{a^2 + b^2} = \sqrt{s^2 + l^2} = l\sqrt{1 + (l/s)^2}/(l/s) = l f(l/s) = lK, \text{ say.}$$

The writer prefers to use $k = s/l$ and to write

$$(II) \quad z = \sqrt{a^2 + b^2} = \sqrt{l^2 + s^2} = l\sqrt{1 + (s/l)^2} = l\sqrt{1 + k^2}.$$

In (1) the function K is tabulated to 4D, and the functions $\tan^{-1}(l/s)$ and $\cot^{-1}(l/s)$ to 0°.001 for $l/s = 1.(001)3(.01)10(.5)30(10)50$.

In (2) the functions $\sqrt{1 + k^2}$ and $\tan^{-1}(s/l) = \tan^{-1}k$ (in radians) are tabulated to 4D, and the functions $\tan^{-1}k$ and $\cot^{-1}k$ to 0°.001 for $k = 0.(001)1$. There is also an 'octant' scheme, with the signs and relative size of a and b as arguments, whereby the phase θ may be readily determined, in conjunction with the table of $\tan^{-1}k$ or $\cot^{-1}k$, wherever it may be in the range 0 to 360°.

The two tables are thus to some extent complementary, in that their arguments cover non-overlapping ranges. Although in most cases there is little to choose between them in practice, it seems to the writer that the tables based on (II) are rather more convenient. (This method is essentially equivalent to the fifth method described in Seeley's letter to *Electrical Engineering*, with a reference to Hudson's *The Engineers' Manual*, p. 257a, b, and to the one described by F. V. Andreae in *Electrical Engineering*, v. 52, p. 724.) This is because, in (II), both s and z are considered as multiples of l , being lk and $l\sqrt{1 + k^2}$ respectively. Thus an ordinary slide rule (or calculating machine) need be set once only, to give multiples of l , and all required values may be read from table and slide-rule with no further resetting. In the tables (1), using (I), however, l is considered as a multiple (the tabular argument) of s , while s is considered as a multiple of l . This means that a slide-rule (or calculating machine) must be set twice instead of once—unless the rule has a scale of reciprocals or something equivalent. The double setting might have been avoided—still with argument l/s —by tabulation of $\sqrt{1 + (l/s)^2} = Kl/s$, so that $z = s\sqrt{1 + (l/s)^2}$. It seems to the writer, however, that in any case $s/l = k$ is a more convenient argument than l/s , because the range is limited to the interval (0, 1). Another advantage of the tables (2) is that z and θ are given on the same page, rather than in separate tables, as in (1).

Besides the two main tables, (1) gives a two-page conversion table (exact) to minutes and seconds of arc for 0(0°.001)1°.

An additional feature of (2) is that it gives a one-page "Table for Reducing Angles to the First Quadrant" for finding the tabular argument in the first quadrant to be used when

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evaluating trigonometric functions of angles up to 9630° or to 224.6239 radians, with a method for extending the latter limit almost to 6400 radians.

J. C. P. MILLER.

EDITORIAL NOTE: W. J. SEELEY, chairman of the department of electrical engineering at Duke University, Durham, N. C., since 1935, is also the author of the following out-of-print item: *Table for the rapid Evaluation of the Square-Root-of-the-Sum-of-the-Squares of two Numbers*, Philadelphia, Engineers' Publ. Co., 1933. 8 p. As in (I) $(a^2 + b^2) = a \cdot K$, $a > b$. The table gives the values of K , for $a/b = [1(.001)3(.01)10(.5)30(10)50; 4D]$, rounded off from 5-place calculations. Mr. Seeley makes the following comments on the above review:

In using the impedance tables with a slide-rule, the rule is first set to divide l by s . In doing this the hair-line is over l . The value of K obtained from the table for this ratio is then placed under the hair-line using the CI, or reciprocal, scale, thus requiring only one setting. Anyone using the table soon learns this procedure. All engineers use slide-rules having the CI, or reciprocal, scale. The original Square-Root table pointed out this procedure in the directions, but it seemed so obvious that it was not deemed necessary when the impedance table was made up.

The chief reason for l/s instead of s/l (which the reviewer prefers) was the greater ease with which the ratios may be read at the bottom of the scale of a slide-rule. Take, for example, $l = 4.5$ and $s = 4.15$. On the rule $s/l = 0.92 + a$ guess; $l/s = 1.08 + a$ guess. The second guess can be made a little more accurately than the first as this rule is a little easier to read at that end. For psychological reasons people seem to prefer to use the lower end of the scales.

240[D, E, M].—FRIEDRICH TÖLKE, *Praktische Funktionenlehre. Erster Band. Elementare und elementare transzendente Funktionen (Unterstufe)*. Berlin, Springer, 1943. Lithoprinted by Edwards Bros., Ann Arbor, Michigan, 1945, vii, 261 p. 19×27 cm. \$6.50. Published and distributed in the public interest by authority of the Alien Property Custodian under license no. A-657.

Impressed by the growing needs of technical engineering and physics for a more extensive tabulation and description of mathematical functions than that provided by the well-known *Tables of Functions* of E. JAHNKE & F. EMDE, the author undertook the production of a six-volume work to meet the new requirements of science. The volume under review was the first of the set. As stated in the preface the second volume was to continue with the elementary functions (higher grade); the third was to contain theta functions; the fourth, elliptic functions; the fifth, hypergeometric and spherical functions; and the sixth, cylindrical functions.

The present volume consists of three parts as follows: I. Definitive differential and integral equations; fundamental properties and corresponding relationships of the elementary and elementary transcendental functions. II. A table of integrals which may be represented by elementary and elementary transcendental functions. III. Tables of the elementary transcendental functions.

The first part (68 pages) begins with a discussion of Gauss's hypergeometric series and its differential equation. Properties are then developed for the exponential function, the logarithmic function, the power function, circular functions and circular functions with the argument $x - \frac{1}{x}$, hyperbolic functions, arc and area functions, the hyperbolic amplitude function and its inverse, trigonometric-exponential and hyperbolic-exponential product functions, and trigonometric-hyperbolic product functions. Properties of the solutions of the following differential system: $u'' \pm au \pm bv = 0$, $v'' \pm av \pm bu = 0$, are then developed. Another section discusses the development of the trigonometric-exponential functions by means of the power function, and the part concludes with a discussion of trigonometric-hyperbolic algebra.

The second part (88 pages) gives an unusually complete set of integrals which may be represented by the elementary and elementary transcendental functions. Some 2040 integrals are included in the list, a number of them being more complicated than those found in standard tables.

The third part (105 pages) of the work is devoted to tables together with an eleven-page description of the functions tabulated and examples illustrating their application. Table 1 gives 5D values, together with differences, over the range $x = 0.(001)1$ of the following functions:

$2\pi x$, $\ln 2\pi x$, $e^{2\pi x}$, $e^{-2\pi x}$, $\sinh 2\pi x$, $\cosh 2\pi x$, $\tanh 2\pi x$, $\coth 2\pi x$, $\text{amp } 2\pi x$, $\sin 2\pi x$, $\cos 2\pi x$, $\tan 2\pi x$, $\cot 2\pi x$, $\sin(2\pi x - \frac{1}{2}\pi)$, $\cos(2\pi x - \frac{1}{2}\pi)$, $\tan(2\pi x - \frac{1}{2}\pi)$, and $\cot(2\pi x - \frac{1}{2}\pi)$.

Table 2 provides 4-6S, mostly 5S, over the range $x = 0.(01)40$ of the following functions:

$\frac{1}{2}\pi x$, $e^{\frac{1}{2}\pi x}$, $e^{-\frac{1}{2}\pi x}$, $\sin \frac{1}{2}\pi x$, and $\cos \frac{1}{2}\pi x$.

Table 3 gives 3-4S values, together with differences, over the range $x = 0.(01)5$ for the following functions:

$\text{Ei}(x)$, $\text{Ei}(-x)$, $\text{Shi}(x)$, $\text{Chi}(x)$, $\text{Si}(x)$, and $\text{Ci}(x)$,

where we use the abbreviations:

$$\text{Shi}(x) = \int_0^x \frac{\sinh t}{t} dt = \frac{1}{2}[\text{Ei}(x) - \text{Ei}(-x)], \text{ and}$$

$$\text{Chi}(x) = \int_1^x \frac{(\cosh t - 1)dt}{t} + \ln x = \text{Shi}(x) + \text{Ei}(-x) = \frac{1}{2}[\text{Ei}(x) + \text{Ei}(-x)],$$

according to Tölke's series, p. 145, 159; see JAHNKE & EMDE, *Tables of Functions*, p. 2. This table of Tölke appears to provide the first set of values ever computed¹ for $\text{Shi}(x)$ and $\text{Chi}(x)$.

Short auxiliary tables are also given as follows:

For $\frac{1}{2}\pi x$, $e^{\frac{1}{2}\pi x}$, $e^{-\frac{1}{2}\pi x}$, $\sin \frac{1}{2}\pi x$, and $\cos \frac{1}{2}\pi x$, over the range $x = [0.001(0.001)0.01; 3 - 5S]$.

For $2\pi x$, $e^{2\pi x}$, and $e^{-2\pi x}$ over the range $x = [0(1)10; 5-13S]$.

For e^x and e^{-x} from 1 to 27S over the range $x = 1(1)50$.

For $p\pi x$, $p = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 2$, over the range $x = [1(1)20; 5D]$, and for $p\pi/x$ over the range $x = [1(1)10; 5D]$.

Values of $m!$ and $1/m!$ for $m = 2(1)10$, values of $m!/(m-n)!$ for $m, n = 1(1)10$, the binomial coefficients, ${}_nC_r$, to $n = 15$, and a number of standard mathematical constants; 10D.

It will thus be seen from what has been said above that the present work is a distinct addition to the literature of mathematical tables. The printing is clear and the descriptive portions contain a wealth of charts which add much to the usefulness of the work.

H. T. D.

¹ EDITORIAL NOTE: Tables of $\text{Shi}(x)$ were given by C. A. BRETSCHNEIDER, (a) *Archiv d. Math.*, v. 3, 1843, p. 34, $x = [1(1)10; 20D]$; also p. 31, $\text{Shi}(1)$ and $\text{Chi}(1)$, each to 35D; (b) *Z. Math. Phys.*, v. 6, 1861, p. 132-139, $x = [1(1)10; 20D]$, $[1(0.01)1(1)7.5; 10D]$. In the same places he gives similar tables of $\text{Chi}(x) = \int_1^x (\cosh t - 1)dt/t + \ln x + \gamma = \frac{1}{2}[\text{Ei}(x) + \text{Ei}(-x)]$, where γ is Euler's constant. In R. So. London, *Trans.*, v. 160, 1870, p. 388, J. W. L. GLAISHER has noted an error in (b), $\text{Shi}(4.9)$ for 18.66679 10435, read 18.66687 10435. The notation $\text{Shi}(x)$ and $\text{Chi}(x)$ seems to have been first adopted by FLETCHER, MILLER, & ROSENHEAD, in their *Index to Mathematical Tables*, London, 1946. It should be noted that the definitions by Tölke of four functions are not in agreement with those so named by any other author. Tölke's values for his $\text{Ei}(x)$, $\text{Ei}(-x)$, $\text{Ci}(x)$, $\text{Chi}(x)$, defined by series, and not by integrals, must each be increased by Euler's constant to get the values used by all other authors of tables for functions so named.

We have reason to believe that the tables listed under (b) above were published by Bretschneider earlier (Easter, 1859) in a Program at the Realgymnasium in Gotha, where he taught; but we have not yet seen a copy.

241[D, U].—W. M. JONES, "Table of direction-cosines to four figures at intervals of one degree in latitude and longitude over a zone of 60° of longitude and latitude from 0° to 40°," *New Zealand J. Sci. and Tech.*, v. 26, section B, 1944, p. 156-159.

"The method of calculating distances between two points on the earth's surface from the direction-cosines of the normals to the surface at these points is well known. It was

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introduced into seismology by H. H. TURNER,¹ and has been further developed by BULLEN, COMRIE, and JEFFREYS,² and tables of geographical and geocentric direction-cosines are available for seismological observatories, so that for a given epicentre the distances to any observatory can be quickly calculated. The tables below are intended for use in seismological (also perhaps navigational and radio-transmission) problems where the distance from a given point to any point inside a certain region may be required. . . . The direction-cosines a , b , c , are given by

$$a = \sin \theta \cos \phi, \quad b = \sin \theta \sin \phi, \quad c = \cos \theta$$

where θ is the co-latitude and ϕ the longitude."

In the tables, p. 156-158, are given, to 4D, the values of a , b , c , for north lat., $0(1^\circ)40^\circ$; east long. $150^\circ(1^\circ)180^\circ$. Appropriate changes of sign give results for south latitudes and west longitudes.

If another point has direction-cosines a' , b' , c' , the angle Δ between the two normals, which in the case of a spherical earth represents the arcual distance along the great circle joining the two points, is given by

$$\cos \Delta = aa' + bb' + cc',$$

or

$$4 \text{ haversine } \Delta = 2 \text{ vers } \Delta = (a - a')^2 + (b - b')^2 + (c - c')^2.$$

With the aid of a table of complete squares, 1000 to 9999 (such as Barlow-Comrie's), and Turner's table (or a table of haversines, see RMT 197), Δ is readily found.

R. C. A.

¹ H. H. TURNER, "On a method of solving spherical triangles, and performing other astronomical computations, by use of a simple table of squares," R.A.S., *Mo. Notices*, v. 75, 1915, p. 530-541. On p. 538-541 is a table of 2 vers Δ , $\Delta = [0(6')4^\circ.9; 6D]$, $[5^\circ(6')59^\circ.9; 4D]$, $[60^\circ(6')119^\circ.9; 3D]$.

² L. J. COMRIE, *The Geocentric Direction Cosines of Seismological Observatories, with Introduction* by Harold Jeffreys, London, B.A.A.S., 1938, viii, 14 p. H. JEFFREYS and K. E. BULLEN, *Seismological Tables*, London, B.A.A.S., 1940, 48 p. + 1 plate.

242[E, L].—C. L. PEKERIS & W. T. WHITE, "Differentiation with the cinema integrator," Franklin Inst., *J.*, v. 234, 1942, p. 21. 15.7×23.7 cm.

Table I gives $F_n(x)$, $G_n(x)$, $n = 1(1)3$, $x = [0(.05).1, .2(.2)6(.5)7.5; 4D]$, where, if $E(x) = x^{-1}e^{-x}$, $F_1(x) = 2xE(x)$, $F_2(x) = E(x)[H_1(x) - \frac{1}{2}H_2(x)]$, $F_3(x) = E(x)[H_1(x) - \frac{1}{2}H_2(x) + (1/32)H_3(x)]$, $H_n(x) = (-1)^ne^{x^2}d^n(e^{-x^2})/dx^n$ (Hermite polynomials), $G_1(x) = -e^{-x}L_1(x)$, $G_2(x) = -e^{-x}[L_1(x) + L_2(x)]$, $G_3(x) = -e^{-x}[L_1(x) + L_2(x) + \frac{1}{2}L_3(x)]$, $L_n(x) = e^{x^2}d^n(xe^{-x^2})/dx^n$ (Laguerre polynomials).

H. B.

243[E, L, M].—CARL E. SMITH, *Applied Mathematics for Radio and Communication Engineers*. New York and London, McGraw-Hill, 1945. viii, 336 p. 13.2×24 cm. \$3.50.

According to the title-page the author is "assistant director, Operational Research Staff, Office of the Chief Signal Offices, War Department; director of engineering (on leave), United Broadcasting Company, Cleveland; president (on leave), Smith Practical Radio Institute, Cleveland."

On p. 249-316 are the following tables and graphs:

8. Tables of exponential and hyperbolic functions in terms of nepers, Tables of e^x , e^{-x} , $\sinh x$, $\cosh x$, $\tanh x$, for $x = 0(.01)1(.05)2.4(.1)5(1)10$, mostly 4 or 5D or S. The tables are taken from E. S. ALLEN, *Six-Place Tables*, see *MTAC*, v. 1, p. 384.

9. Table of exponential and hyperbolic functions in terms of decibels: If N decibels = x nepers, $N/x = 10 \log (P_1/P_2)/.5 \ln (P_1/P_2) = 20/\ln 10 = 8.68588964$; or $x/N =$

.11512925. For $N = .05, 1(1)2(2)4(.5)20(1)50(5)100, 2.5, 3.5$, are given values of x , $e^x = k$, e^{-x} , $e^{\pm x}$, $\sinh x = \frac{1}{2}(k^2 - 1)/k$, $\cosh x = \frac{1}{2}(k^2 + 1)/k$, $\tanh x$, $\tanh(\frac{1}{2}x) = (k - 1)/(k + 1)$, to 5 or 6S. This appears to be the first published mathematical table in terms of decibels.¹

10. Curves and tables of Bessel Functions (p. 255-293), Graphs of $J_n(x)$, $n = 0(1)20$, $0 < x < 20$; maxima and minima, to 4D, are marked. Tables of $J_0(x)$, $-J_1(x)$, for $x = [0(.01)15.5; 6D]$, are adapted from Meissel's table in GRAY, MATHEWS & MACROBERT, *Bessel Functions*, London, 1931, and hence have the same errors in $J_0(.62)$, $J_1(.787)$; see *MTAC*, v. 1, p. 290 and 298. Tables of $J_n(x)$, $x = 1(1)29$, mostly 4S; for $x = 1(1)5$, $n = 0(.5)7(1)27$; $x = 6(1)10$, $n = 0(.5)14(1)37$; $x = 11(1)15$, $n = 0(.5)19(1)46$; $x = 16(1)20$, $n = 0(.5)19(1)54$; $x = 21(1)24$, $n = 0(.5)7(1)60$; $x = 25(1)29$, $n = 0(1)44$. These tables and those of paragraph 11 are copied (with permission of the Alien Property Custodian) from JAHNKE & EMDE, *Tables of Functions*, 1943, with the error in $J_0(21)$, which is corrected in the 1945 edition; see *MTAC*, v. 1, p. 109.

11. Curves and tables of sine and cosine integrals (p. 294-298), Graphs of $\text{Si}(x)$ and $\text{Ci}(x)$, $0 < x < 15$. Tables, from JAHNKE & EMDE of $\text{Si}(x)$, $\text{Ci}(x)$, $x = [0(.01)1(1.1)5(1)15(5)100(10)200(100)10^6]$; mostly 4D], 10^{100} 4D for $\text{Si}(x)$ and 1-3S for $\text{Ci}(x)$. The error in $\text{Si}(65)$ has been preserved; see *MTAC*, v. 1, p. 395.

12. Factorials of numbers from 1 to 20 and their reciprocals.

22. Wave forms (p. 311-316), Various Graphs: (a) Square sine wave, (b) Square cosine wave, (c) Sawtooth sine wave, (d) Sawtooth cosine wave, (e) Positive scanning wave, (f) Negative scanning wave, (g) Half-wave rectifier pulse wave, (h) Full-wave rectifier pulse wave, (i) Unsymmetrical sawtooth wave, (j) Scanning wave with β flyback time, (k) Rectangular pulse wave, (l) Symmetrical trapezoidal wave, (m) Symmetrical triangular pulse wave, (n) Fractional cosine pulse wave.

R. C. A.

¹The terms "decibel" and "neper" were adopted about 1929; see W. H. MARTIN, "Decibel—the name for the transmission unit," *Bell System Technical J.*, v. 8, 1929, p. 1-2. A decibel, one tenth of a bel, is a unit in communication engineering, acoustics, and allied fields. This unit is defined by the statement that two amounts of power, P_1 , P_2 , differ by one transmission unit when they are in the ratio of 10^{-1} , and any two amounts of power differ by N transmission units when they are in the ratio $10^{N(10)}$ or $N = 10 \log(P_1/P_2)$. The unit bel was derived from the name of ALEXANDER GRAHAM BELL (1847-1922). The neper, derived from the name of JOHN NAPIER or NEPER (1550-1617), is also a transmission unit with basic power ratio e^2 , so that here $e^{2x} = P_1/P_2$ or $x = .5 \ln(P_1/P_2)$.

244[E, M].—A. I. NEKRASOV, "O dvizhenii tiazhelykh tel pri kvadraticnom zakone soprotivleniia" [Motion of heavy bodies in a medium in accordance with the quadratic law of resistance], *Prikladnaia Matematika i Mekhanika (Applied Mathematics and Mechanics)*, v. 9, May, 1945, p. 197-206, table p. 204-205. 17×25.8 cm.

The main section of this paper "treats of the motion of heavy bodies vertically, from great heights. If the usual law of the distribution of density in accordance with the height of the atmosphere is accepted, the problem will be reduced to quadratures, which can be calculated only approximately." By introducing a new transcendental function and the following tables, more exact results may be obtained:

$$f_1(z) = p e^{-p} \int_0^z e^{x^2} (x^2)^p dx, \\ f_2(z) = 1 - e^{-p},$$

where $p = 1/n + 1$, $n = 4.256$, $x = [0(.01)1.5(1)5(2)10(.5)20(1)38(2)50; 5D]$.

245[F].—HANSRAJ GUPTA, "Congruence properties of $\tau(n)$." *Benares Math. Soc., Proc.*, s. 2, v. 5, 1943, p. 17-22. 18.4×25.3 cm.

This article contains (p. 22) a one-page table of Ramanujan's $\tau(n)$ for $n = 1(1)130$. This function, which is important in the theory of elliptic modular functions and their

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Fourier coefficients, is defined and discussed in *MTAC*, v. 1, p. 183-4, where a review is given of a previous table for $n = 1(1)300$. A comparison of the two tables has been made. They are in complete agreement. The author gives no indication of his method of calculation. Ramanujan's original table¹ was for $n = 1(1)30$.

D. H. L.

¹ S. RAMANUJAN, "On certain arithmetical functions," *Cambridge Phil. So. Trans.*, v. 22, no. 9, 1916, p. 174; *Collected papers of Srinivasa Ramanujan*, Cambridge, 1927, p. 153. In copying $\tau(17)$ Gupta omitted the "-" sign.

246[F].—ERNST S. SELMER & GUNNAR NESHEIM, "Eine neue hypothetische Formel für die Anzahl der Goldbachschen Spaltungen einer geraden Zahl, und eine numerische Kontrolle," *Archiv für Math. og Naturvid.*, v. 46, 1943, p. 1-18. 15.7×23.6 cm.

This note, concerning the famous unsolved Goldbach problem, gives numerical evidence as to the comparative accuracy of three similar formulas for the number $G(2n)$ of ways that the number $2n$ is the sum of two primes. These formulas are all somewhat conjectural and are due to Stäckel,¹ Shah & Wilson², and the authors. The authors calculate not the function $G(2n)$ but its sum, $\sum G(2n)$, taken over all even numbers from 10^4 to $10^4 + 100$, and obtain 312778 as the total number of ways that the 51 even numbers in this range are the sums of two primes. This is compared with $51 \cdot G(10^4 + 50)$ as given approximately by the formulas referred to above. The results are respectively 314275, 312442 and 313918. In the opinion of the reviewer these results are not very conclusive, since this indicates in a rough way the behavior of $G(2n)$ only very near to $2n = 10^4$. Near $2 \cdot 10^4$ or 10^7 quite different results of comparison might be expected. Page 11 contains a table giving for each of the 51 even numbers $2n$ referred to above, the value of the product

$$\frac{\leq \sqrt{2n}}{\prod_{p|2n} [(p-1)/(p-2)]}, \quad p > 2.$$

This is used to show that for this range of $2n$ a certain "correction factor" suggested by Brun³ is, on the average, very close to unity and so can be neglected.

D. H. L.

¹ P. STÄCKEL, "Die Lückenzahlen r -ter Stufe und die Darstellung der geraden Zahlen als Summen und Differenzen ungerader Primzahlen," *Heidelberger Akad. d. Wissen., Sitzungsab., math.-natw. Kl.*, 1917, no. 15, 52 p.

² N. M. SHAH & B. M. WILSON, "On an empirical formula connected with Goldbach's theorem," *Cambridge Phil. So., Proc.*, v. 19, 1919, p. 238-244.

³ V. BRUN, "Über das Goldbachsche Gesetz und die Anzahl der Primzahlpaare," *Archiv für Math. og Naturvid.*, v. 34, no. 8, 1916, 19 p.

247[F].—ERNST S. SELMER & GUNNAR NESHEIM, "Eine numerische Untersuchung über die Darstellung der natürlichen Zahlen als Summe einer Primzahl und einer Quadratzahl," *Archiv für Math. og Naturvid.*, v. 47, 1943, p. 21-39. 15.7×23.6 cm.

This paper contains tables of the function $R(n)$, the number of ways that n is the sum of a square and a prime. Table I (p. 30-31) gives $R(n)$ for $n \leq 2000$, while Table II [p. 36-37] gives $R(n)$ for $1010000 \leq n \leq 1010100$. These latter values are compared with the approximation

$$R(n) \sim \sqrt{n}/\ln(4n/\pi^2) \cdot \prod_{2 < p < \sqrt{n}} [1 - (n/p)/(p-1)]$$

where (n/p) is the Legendre symbol.¹

D. H. L.

¹ On p. 35 the authors refer to an error in D. N. LEHMER, *List of Prime Numbers*, Washington, 1914; p. 11, col. 13, for 8151, read 8051. This error was already noted in my *Guide to Tables in the Theory of Numbers*, Washington, 1941, p. 161, and in *Scripta Mathematica*, v. 4, 1936, p. 198.

248[F].—CARL STØRMER, "Sur un problème curieux de la théorie des nombres concernant les fonctions elliptiques," *Archiv for Math. og Naturvid.*, v. 47, 1943, p. 83-85. 15.7 × 23.6 cm.

In studying the solutions of the equation

$$(1) \quad m \tan^{-1}(1/x) + n \tan^{-1}(1/y) = k\pi/4,$$

where x, y, m, n, k are to be integers, Størmer found¹ in 1895, four sets of solutions,² other than the trivial solutions $x = y = 0$ or 1. More recently LJUNGGREN gave similar solutions³ for the equation

$$(2) \quad m \tan^{-1}(\sqrt{D}/x) + n \tan^{-1}(\sqrt{D}/y) = k\pi/6,$$

and found 14 solutions when $D = 3$, and 10 when $D = 2$. Writing (1) in the form

$$m \int_x^\infty \frac{dt}{(1+t^2)} + n \int_y^\infty \frac{dt}{(1+t^2)} = k(\pi/4),$$

Størmer generalized it to the case of elliptic integrals

$$(3) \quad m \int_x^\infty \frac{dt}{\sqrt{4t^3 - g_2t - g_3}} + n \int_y^\infty \frac{dt}{\sqrt{4t^3 - g_2t - g_3}} = k\omega,$$

where g_2 and g_3 are integers, and ω is the real semi-period of the corresponding Weierstrass function, $\wp(u)$. Considering the case $g_2 = 252$, $g_3 = -648$, $4t^3 - g_2t - g_3 = 4(t-6)(t-3)$

$(t+9)$ and $\omega = \int_0^\infty \frac{dt}{\sqrt{4t^3 - g_2t - g_3}} = .582808\dots$; 9 sets of integral values of $x, y, m,$

n, k are given in a little table of 9 solutions of (3), of which the author notes the last one as the most interesting, namely:

$$3 \int_{11}^\infty \frac{dt}{\sqrt{4t^3 - 252t + 648}} + 2 \int_{13}^\infty \frac{dt}{\sqrt{4t^3 - 252t + 648}} = \omega.$$

R. C. A.

¹ C. Størmer, "Solution complète en nombres entiers m, n, x, y, k de l'équation $m \arctg(1/x) + n \arctg(1/y) = k(\pi/4)$," *Norsk Videnskabs Akademi, Christiania, Skrif-ter, math.-naturw. Kl.*, 1895, no. 11; see also his "Sur l'application de la théorie des nombres entiers complexes à la solution en nombres rationnels $x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_n, k$ de l'équation: $c_1 \arctg x_1 + c_2 \arctg x_2 + \dots + c_n \arctg x_n = k\pi/4$," *Archiv for Math. og Naturvid.*, v. 19, no. 3, 1896, 96 p. A reference may be given to a third paper by this author, "Solution complète en nombres entiers de l'équation $m \arctg(1/x) + n \arctg(1/y) = k\pi/4$," *So. Math. d. France, Bull.*, v. 27, 1899, p. 160-170.

² All four of the solutions found by Størmer ($k = 1$ in each case) were known in the eighteenth century, namely: $m = 2, n = -1, x = 2, y = 7$, given by J. Hermann in a letter of 1706 to Leibniz; $m = 2, n = 1, x = 3, y = 7$, by Vega in 1794; $m = n = 1, x = 2, y = 3$ by Euler in 1738; $m = 4, n = -1, x = 5, y = 239$ by Machin in 1706.

³ W. LJUNGGREN, "Über einige Arcustangensgleichungen die auf interessante unbestimmte Gleichungen führen," *Arkiv f. Mat., Astron. o. Fysik*, v. 29A, no. 13, 1942, 11 p.

249[H].—ANDRES ZAVROTSKY (1904—), *Tablas para la Resolución de las Ecuaciones Cúbicas*, Caracas, Venezuela, Editorial Standard, 1945. vi, 163 p. 16 × 23.5 cm. 5 bolivares. American agent: G. E. Stechert & Co., New York. \$2.00.

In *MTAC*, v. 1, p. 441f, we referred to tables for solving cubic equations by NOGRADY, KATZ, JAHNKE & EMDE, and GUNDELFINGER. Six other references to tables may be given as follows:

In J. H. LAMBERT, (a) *Zusätze zu den logarithmischen und trigonometrischen Tabellen* . . . , Berlin, 1770, p. 163-173; (b) *Supplementa Tabularum Logarithmicarum* . . . ed. by A. FELKEL, Lisbon, 1798, p. 151-161; is Table XXIX, "Equationes cubicae radicum

realium," for solving equations $|x^3 - x| = a$. For $x = 0.(001)1.155$, the values of a (varying from 0 to .3857989) are given to 7D. There are illustrative examples.

In P. Barlow, *New Mathematical Tables*, London, 1814, p. 186-191, there is Table IV ("for the solution of the irreducible case in cubic equations") of $x^3 - x = a$, for $x = 1.(0001)1.1549$, giving values of a to 8D. Explanatory text and illustrations are added.

Furthermore, is the extensive monograph of J. P. KULIK, "Beiträge zur Auflösung höherer Gleichungen überhaupt und der kubischen Gleichungen insbesondere," Prague, 1860, p. 21-123, 23 × 29 cm.; in Česká Společnost Náuk, Prague, *Abhandlungen*, s. 5, v. 11. On p. 58-123 is a table of $|x^3 \pm x| = a$, 6-8D, with Δ , for $x = 0.(0001)3.28$. The equation has three real roots and (i) the smallest is given, $x = 0.(0001).5999$, the 8-place part of the table, occurring here for $x = .55+$, fills p. 69; (ii) the middle root is given $x = .6.(0001).9999$; (iii) the largest root is given $x = 1.(0001)1.1547$. When the equation has only one real root, it is given, $x = 1.1547.(0001)3.28$, and a varies from .384899 to 32.007552; various four-term cubic equations are solved by this table.

In an elaborate discussion by FERDINAND KERZ, "Ueber die Beurtheilung der Wurzeln einer vorgelegten cubischen Gleichung," *Archiv Math. Phys.*, v. 41, 1864, p. 68-102; v. 42, 1864, p. 121-179, 240, 482f; v. 44, 1865, p. 1-49, 129-183, 379-440; are several tables. T. III, v. 44, p. 5-38, gives the values of $x + x^3 = a$, for $x = [0.(001)1(.01)10(.1)24; 7-10S]$, Δ , a varying from 0 to 13848; and T. IV, p. 133-174, the values of $|x - x^3|$, $x = [0.(001)1.2(.01)10(.1)24; 7-9S]$, Δ .

In ALSTON HAMILTON, "The irreducible case of the cubic equation," *Annals Math.*, s. 2, v. 1, Oct. 1899, p. 45, there is a table of $x^3 - x$, $x = [1.(001)1.16; 6D]$.

And finally, there are tables of $x \pm x^3$, $x = [0(.01)1(.05)2(.1)7; 7D]$ in R. HEGER, *Fünfstellige logarithmische und goniometrische Tafeln, sowie Hilfstafeln zur Auflösung höherer numerischer Gleichungen*, second improved ed., Leipzig, 1913, p. 76f.

Thus, to aid in their solutions of a cubic equation Lambert, Barlow, Kulik, Katz, Kerz, Hamilton and Heger tabulate functions of the form $x^3 \pm x$. Kulik does not refer to either Lambert or Barlow.¹ Kerz, Hamilton, and Heger make no reference to any other table.

The work under review, published in January, 1945, is in some respects the most detailed set of tables for the solution of cubic equations yet published. It is the first to give complex roots. If accurate, these tables ought to be, generally, also the most useful. They give, to 5D, roots of all equations

$$(1) \quad x^3 + px + q = 0,$$

for which $p = -100(1) + 100$, $q = 0(1)100$.

For applying the table directly, it is supposed that q is always a positive integer. If it happens that a negative sign precedes q , a new equation is formed whose roots are the negative of those in the original equation; the sign before q is then positive and the sign before p is unchanged.

Since the discriminant of (1) is

$$(2) \quad -4p^3 - 27q^2$$

it is obvious that when p and q are both positive, two of the roots of equation (1) are imaginary. Thus on pages 2-81 are given the real and imaginary parts of these roots, on opposite pages, $p = 0(1)100$, $q = 0(1)100$, four pages being required for each of the groups of values of p , 0(1)5, 6(1)10, ... 95(1)100. Each real root of an equation in this case is clearly the negative of twice the real part of a complex root.

Again, if p is negative, or zero, the discriminant indicates that there may still be two imaginary roots only if $-p \leq 3[(q/2)^{1/3}]$ or $q \geq 2[(-p/3)^{1/3}]$. On p. 162 is a five-place "table of critical values," giving $2[(n/3)^{1/3}]$, $n = 0(1)40$, and $3[(n/2)^{1/3}]$, $n = 0(1)100$, by means of which it may be at once determined, for a given pair of values of p and q , whether there are complex roots or not. From this table it is obvious that all imaginary roots are

found when $-p < 41$. Thus, for each pair of values (p, q) , $p = -100(1) - 41$, $q = 0(1)100$, the roots of (1) are real; the two positive roots are given, p. 117-161. For other values of $-p$ there are certain values of q for which the roots are complex; for example, $-p = 40$, $q = 98(1)100$, or $-p = 2$, $q = 2(1)100$. For $-p = -40(1)0$, $q = 0(1)100$, the two complex roots, or the two positive roots, as the case may be, are clearly set forth, p. 82-117.

The author gives no indication as to how he calculated his table. He does add, however, "If p (or) q are fractional, use interpolation by differences. If the numerical value of p and (or) q slightly exceeds 100 extrapolation by differences is still admissible; but for still greater values of the coefficients, perform the substitution $x = 10y$, or $x = 100y$ or generally $x = ny$, where n may be any power of 10, or any other number."

If we consider the general form of the cubic equation, $as^3 + bs^2 + cs + d = 0$, and substitute $s = x - b/(3a)$, we are led to equation (1), if $p = (3ac - b^2)/3a^2$, $q = (27ad - 9abc + 2b^3)/27a^3$.

We are told that the present work is the first of a series forming a *Thesaurus Aequationum*, by analogy with the *Thesaurus Logarithmorum* of Vega; that the second, third and fourth parts "Logaritmos de las raíces de las ecuaciones cúbicas," "Raíces de las ecuaciones cuadráticas," "Tablas para la formación de las ecuaciones cúbicas," are ready for publication, and that the fifth, "Raíces de las ecuaciones quinticas," is in the course of preparation. One of those to whom the author expresses indebtedness for revision of his manuscript is F. J. DUARTE, the well-known table-maker (1925-1933), and president of the Venezuelan Academy of Physical, Mathematical, and Natural Sciences.

R. C. A.

¹ It may be mentioned that on p. 25 of the Kulik's monograph is the following footnote (freely translated): "I possess a manuscript, which is a continuation of the Burckhardt table, from 3 million to 100 million, on 4212 closely written folio pages. To friends of science this manuscript is available both for inspection, and also for copying any part of the same. One million fills only 44 pages." [The reason why $43\frac{1}{2}$ would be more accurate than 44 may be shown later.] Kulik died in 1863, three years after writing this. In 1867 the great table, in 8 manuscript volumes, was deposited in the Vienna Academy of Sciences; in 1911 the second of these volumes was missing. D. H. L. has a photostat copy of the table extending from 9 000 000 to 12 642 600, that is, somewhat more than the last third of v. 1. For further information about this ms. see D. N. LEHMER, *Factor Table for the First Ten Millions*, Washington, 1909, cols. IX, X, XII-XIV; also D. N. LEHMER, *List of Prime Numbers* from 1 to 10 006 721. Washington, 1914, cols. XI, XII.

250[H, L].—MANUEL SANDOVAL VALLARTA, "Nota sobre las raíces de algunas ecuaciones trascendentes," Sociedad Matem. Mexicana, *Bol.*, v. 2, Jan.-Apr., 1945, p. 13-14. 16.5 × 23 cm.

The author states that in certain problems of physics he was obliged to calculate the zeros of certain transcendental functions, and that these values are now made available to other investigators. Four types of functions are considered.

I. *Hermite polynomials*, H_n . $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, $H_5(x) = x^5 - 10x^3 + 15x$, $H_6(x) = x^6 - 15x^4 + 45x^2 - 15$, $H_7(x) = x^7 - 21x^5 + 105x^3 - 105x$. The zeros are given to 5D. These are only a small part of those given to 6D by E. R. SMITH in 1936; see *MTAC*, v. 1, p. 152 f. There are three slight Vallarta errors to be noted when compared with the correct results of Smith: (a) H_4 , S 2.334414, V 2.33442; (b) H_5 , S 2.856970, V 2.85696; (c) H_7 , S 1.154405, V 1.15440.

II. $[e^{-\frac{1}{2}x^2}/(4\sqrt{2\pi})]H_4(x) - J_4(x)$. The first zero is given as .7410.

III. $Ci(x) - J_0(x)$. The first two zeros are given as 1.572 and 4.955.

IV. $J_0(x) - xM(1, \frac{1}{2}, x)$, where $M(1, \frac{1}{2}, x) = 1 + 2x + (4/3)x^2 + (4/3)(2/5)x^3 + (4/3)(2/5)(2/7)x^4 + \dots$. The zero is given as .43712.

Only results are stated; there is no suggestion as to the methods of their derivation.

R. C. A.

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251[I].—A. N. LOWAN & H. E. SALZER, "Table of coefficients for numerical integration without differences," *J. Math. Phys.*, v. 24, Feb., 1945, p. 1-21.

This set of tables furnishes coefficients $B_k^{(n)}(p)$ for use in the formula

$$(1) \quad \int_{x_0+rh}^{x_0+h} f(x)dx = h \sum_{i=\mu-n+1}^n \{B_i^{(n)}(s) - B_i^{(n)}(r)\} f(x_0 + ih) + R_n(f)$$

where $\mu = [n/2]$. The polynomials $B_k^{(n)}(p)$ are, of course, not defined by this relation. An explicit definition may be given by

$$(2) \quad B_k^{(n)}(p) = (-1)^{k+n} \int_0^p \prod_{\lambda=\mu-n+1}^{\mu} (t-\lambda) dt / \{(k+n-\mu-1)!(\mu-k)!\}$$

where the prime on the product sign indicates that the factor $t-k$ is to be omitted. The relation (1) is the result of term by term integration of the Lagrange interpolation formula for equal intervals h . The tables of $B_k^{(n)}(p)$ are for $n = 3(1)7$. The variable p ranges in each case from $\mu - n + 1$ to μ at intervals of .01 for $n = 3, 4, 5$, and at intervals of .1 for $n = 6, 7$. All values are given to 10D. It is easily verified from (2) that if n is odd,

$$B_{-k}^{(n)}(-p) = -B_k^{(n)}(p).$$

This symmetry has permitted a compact arrangement of the tables for $n = 3, 5, 7$.

The table is not illustrated nor is there any discussion of the remainder term $R_n(f)$. To check that he has entered the table correctly and has correctly formed the appropriate differences $B_i^{(n)}(s) - B_i^{(n)}(r)$ which appear in (1), the computer should first form their sum before proceeding to use them as coefficients of the n values of f . This sum should be $s - r$, as may be seen at once by taking $f(x) = 1$ in (1). This procedure is especially important when the values of the B 's have been obtained by interpolation. These tables should prove useful in those cases where the intervals .01 and .1 are not too large.

D. H. L.

252[I, L].—GREAT BRITAIN, H.M. NAUTICAL ALMANAC OFFICE, *Zeros of Laguerre Polynomials and the corresponding Christoffel Numbers*, Department of Scientific Research and Experiment, Admiralty Computing Service, June, 1945. No. SRE/ACS 82 (w). 7 p. 20 × 33 cm. This publication is available only to certain Government agencies and activities.

The Christoffel numbers $\lambda_i^{(n)}$ are the coefficients in the formula of approximate quadrature

$$\int_0^\infty f(x)w(x)dx = \sum_{i=1}^n \lambda_i^{(n)} f(x_i^{(n)})$$

in which the numbers x_i are the n roots of a polynomial $W(x)$ which satisfies an orthogonal relation of type

$$\int_0^\infty x^m w(x) W_n(x) dx = 0 \quad (m = 0, 1, \dots, n-1).$$

When $w(x) = e^{-x}$ the polynomial $W_n(x)$ is the polynomial of Laguerre $L_n(x)$ defined by the equation

$$L_n(x) = e^x (d^n/dx^n) (x^n e^{-x}).$$

The roots $x_i^{(n)}$ are given to 8D for $n = 1(1)10$, the tabular values being thought to be correct to the last figure given. The Christoffel numbers are given to 8S and it is believed that the last figure is generally correct, the error never being more than one unit.

H. B.

253[K].—KARL STUMPF, *Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung*. Berlin, Springer, 1939. Lithoprinted by Edwards Bros., Ann Arbor, Michigan, 1944, vii, 174 p. 20 × 27 cm. \$5.25. Published and distributed in the public interest by authority of the Alien Property Custodian under license no. A-212.

This work was prepared by the author to provide tabular assistance in applying the methods of harmonic analysis developed in his treatise, *Grundlagen und Methoden der Periodenforschung*, Berlin, Springer, 1937. Although the volume is devoted principally to tables, the second part (p. 133–172) contains a number of numerical examples illustrating the application of the tables.

Since the problems with which harmonic analysis deals are concerned entirely with sine and cosine series of the form

$$y = a_0 + a_1 \cos v\alpha + a_2 \cos 2v\alpha + \cdots + a_n \cos nv\alpha \\ + b_1 \sin v\alpha + b_2 \sin 2v\alpha + \cdots + b_n \sin nv\alpha,$$

where $\alpha = 2\pi/(2n+1)$, it is clear that the principal need is for tables which give multiples of the sine and cosine functions for various arguments. It is to provide for this need that the present work has been prepared.

On the assumption that most data subject to harmonic analysis do not require unusual exactness, the tables are computed for the most part to three or four significant figures. They include the following:

Tables 1a, 1b, and 1c give the values of $v\alpha$ (in hundredths of a degree), $|\cos v\alpha|$, and $\sin v\alpha$, where $\alpha = 360^\circ/p$, $v = 0(1)\frac{1}{2}(p-1)$, and p is over the ranges: $p = 2n-1$, $n = 2(1)20$; $p = 4n-2$, $n = 1(1)10$; $p = 4n$, $n = 1(1)10$.

Table 2a gives the product $10x \cos \theta$ over the range $x = 0(1)1009$ and $\theta = 15^\circ, 22\frac{1}{2}^\circ, 30^\circ, 45^\circ, 67\frac{1}{2}^\circ, 75^\circ$.

Table 2b provides values of the product $x \sin \theta$ (or $x \cos \theta$) for $x = 1(1)100$ and $\theta = 1^\circ(1^\circ)90^\circ$.

Table 3a gives the product $x \sin v\alpha$ and $x \cos v\alpha$, $x = 2(2)100$, $p = 21$ to 39 inclusive, excepting integers divisible by 4, and $v = 1, 2$, etc. for all values such that $v\alpha \leq 90^\circ$ if p is even and $v\alpha \leq 180^\circ$ if p is odd.

Table 3b provides the values of the products $x \sin v\alpha$ and $x \cos v\alpha$, $x = 1(1)100$, $p = 20(4)40$, $v = 1, 2$, etc., such that $v\alpha \leq 90^\circ$.

Table 4a gives the values of $h = (a^2 + b^2)^{1/2}$ and $\psi' = \arctan(a/b)$, expressed in grades (g), for $a, b = 1(1)50$, where $100^g = 90^\circ$.

Table 4b is similar to the preceding except that the values of $\alpha \leq \beta$ and $h = (\alpha^2 + \beta^2)^{1/2}$ are given in terms of $\psi' = 1^\circ(1^\circ)50^g$ and $\beta = 51(1)100$.

Tables 6a and 6b contain values respectively of a^2 for $a = .1(1)100.9$ and, to 4S, of \sqrt{a} for $a = 1(1)1009$ and $.1(1)100.9$.

Table 6c evaluates, from 3 to 5D, the functions

$$a, \sin a, (1/a) \sin a, a/\sin a, (a/\sin a)^2, 1/a,$$

where $a = \pi x/200$, as follows:

The first five functions over the range $x = 1(1)100$, the first, third, and fourth for $x = 10(1)200$, the first and third for $x = 20(2)2000$; the third for $x = 2000(10)8000$, and the last for $x = 8100(100)12000$.

Table 6d gives the values of $k \sin x$ and $k \cos x$ to 1D, for $k = 1(1)30$, $x = 2^\circ(2^\circ)100^g$, where $100^g = 90^\circ$.

Table 6e provides 4D values of the following functions:

$$a^\circ, \sin a, \tan a, \sin^2 a, \sin 2a, \cos 2a, 1/(2 \sin a), 1/(4 \sin^2 a),$$

over the following ranges:

All the functions for $a = 1^\circ(1^\circ)50^g$; $\tan a$ for $2a = -1^\circ(2^\circ)199^g$; and all the functions except $\sin 2a$ and $\cos 2a$ for $a = 51^\circ(1^\circ)100^g$.

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Table 6f gives 4D values of the functions

$$1/\sqrt{n}, 1/\sqrt{10n}, (\pi/n)^{1/2}, [\pi/(10n)]^{1/2},$$

for $n = 1(1)100$; it also includes 5D values of e^{-x} for $x = .1(1)10$, and 4 or 5D or S values of e^{-x^2} , e^{x^2} , $e^{-1/2x^2}$, $e^{1/2x^2}$ for $x = .1(1)5$.

Tables 1d and 5, which we have omitted above, give special schemes pertaining to harmonic analysis. Table 1d also provides numerical values for $360^\circ/n$, $n = 1(1)180$.

The work concludes with a bibliography on harmonic analysis, as in the original edition, and on the tables previously available for an application of the various methods. The reviewer, in a previous article, *MTAC*, v. 1, p. 193, has listed the tables especially adapted for statistical work in this field. The lithoprint copy is clearly printed and strongly bound.

H. T. D.

254[L].—W. G. BICKLEY, "Notes on the evaluation of zeros and turning values of Bessel functions.—III. Interpolation by Taylor series," *Phil. Mag.*, s. 7, v. 36, Feb., 1945, p. 131–133. 17×25.4 cm.

The remark that the derivatives of $J_n(x)$ can be expressed in terms of differences of functions $J_{n+p}(x)$ can be used in combination with a table of Bessel functions $J_n(x)$ with intervals of 0.1 to compute zeros of $J_n(x)$. This is illustrated by a numerical example with $n = 20$ and a zero close to 30. In the short table the values of $J_n(30)$ are given first for $n = [16(2)24; 13D]$ with 3 values of δ^2 and one of δ^4 centered around $n = 20$. They are given next for $n = [15(2)25; 13D]$ with the 5 values of δ , 3 of δ^2 and one value of δ^4 needed for the formation of the Taylor series of $J_n(30 - h)$. The first approximation $J_n(30) = 0$ is corrected by including terms up to h^3 in the equation $J_n(30 - h) = 0$, the method being essentially that of EDMOND HALLEY.¹ In the next approximation linear interpolation is used and so the root of the preceding equation is calculated to 5D. A typical computation is given and 13 decimals are kept but it is thought that they are not all reliable.

H. B.

¹ H. Bateman, "Halley's methods for solving equations," *Amer. Math. Mo.*, v. 45, 1938, p. 11–17.

EDITORIAL NOTE: The values of $J_n(30)$, $n = 15(1)25$ are in agreement, so far as they go, with those given to 35D, by K. HAYASHI, *Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen*, Berlin, 1930, p. 56.

255[L].—W. G. BICKLEY & J. C. P. MILLER, "Notes on the evaluation of zeros and turning values of Bessel functions.—II. The McMahon series," *Phil. Mag.*, s. 7, v. 36, Feb., 1945, p. 124–131. 17×25.4 cm.

The coefficients from A_1 up to A_{13} are given in the asymptotic expansion of the McMahon type

$$c_{n,s} \sim \beta - \sum_{n=0}^{\infty} A_{2n+1} \beta^{-(2n+1)} 2^{-2n-2} / (2n+1)!$$

for the zero $c_{n,s}$ of $J_n(x) \cos \alpha - Y_n(x) \sin \alpha$, where $\beta = (s + \frac{1}{2}n - \frac{1}{2})\pi - \alpha$, α is a real angle in circular measure and s is a fairly large positive integer. A notable simplification in the numerical coefficients is obtained by expressing them in terms of a new parameter $\lambda = (\mu - 1)/24$ which is particularly useful when n is half an odd integer. The series in descending powers of β are given with the exact numerical coefficients in the cases $n = 2$, $n = 20$, $n = 1\frac{1}{2}$, $n = 2\frac{1}{2}$. In the last two cases the expansion is carried as far as the term in β^{-10} , the transcendental equations being respectively $\tan(x + \alpha) = x$ and $\tan(x + \alpha) = 3x/(3 - x^2)$. In the latter case the series checks with the general series for roots of $\tan x = x/(a - cx^2)$, given *MTAC*, v. 1, p. 273, but in the last term of the general series the coefficient of x^7 seems to be too large. The explanation is that the term $23c$, in line 13, should be $23c/15$. The details of the application of the McMahon series to the numerical

application of zeros are given, the quantity β being replaced by a second quantity $\alpha = 4\beta/\pi$ and the series written in the form

$$c_{n,s} \sim \frac{1}{2}\pi\beta - \frac{B_1}{\alpha} - \frac{B_1 \cdot B_2}{\alpha \cdot \alpha^3} - \frac{B_1 \cdot B_2 \cdot B_3}{\alpha \cdot \alpha^3 \cdot \alpha^5} - \dots,$$

Numerical values of the coefficients B_r , $r = [1(2)13; 12D \text{ to } 4D]$, are given in Tables I and II, for $n = 1(1)20$ and $2n = 1(2)21$, $3n = 1, 2, 4, 5$; $4n = 1(2)7$. Details of the calculation are given for $n = 2$.

H. B.

256[L].—W. G. BICKLEY & J. C. P. MILLER, "Notes on the evaluation of zeros and turning values of Bessel functions. V. Checks," *Phil. Mag.*, s. 7, v. 36, Mar. 1945, p. 206-216. 17×25.4 cm.

Since the McMahon expansions have the same coefficients for $J_n(x)$ and for $Y_n(x)$, but with different arguments, it is possible to arrange the zeros $j_{n,s}$, $y_{n,s}$ of J_n and Y_n in a single sequence for each value of n . This is exemplified by a table on p. 207 in which $n = 0(1)3$ and from 5 to 7 early zeros of each function are given to 5D. A method of checking by means of differences is explained.

H. B.

257[L].—G. P. DUBE, "Electrical energy of two cylindrical charged particles," *Indian J. Physics*, v. 17, 1943, p. 192. 17.5×25.5 cm.

The equation

$$2P[I_0(t)K_2(t) + I_1(t)K_1(t)] = x[K_0(x)/K_1(x)]$$

is solved numerically, and the root x is given to 2D, for $t = .1, .5(.5)3$.

H. B.

258[L].—G. P. DUBE & S. N. JHA, "On the theory of emission of alpha-particles from radioactive nuclei," *Indian J. Physics*, v. 17, 1943, p. 354. 17.5×25.5 cm.

Bessel functions of imaginary index and real argument are used, and to facilitate some computations, a table is given of the function

$$g(x) = \cos^{-1}(x^{\frac{1}{2}}) - x^{\frac{1}{2}}(1 - x)^{\frac{1}{2}}, \text{ for } x = [.12(.01).3; 4D].$$

H. B.

259[L].—ERWIN FEHLBERG, "Eine Bemerkung zur numerischen Differenziation durch Approximation, ausgeführt am Beispiel der Kugelfunktionen als Approximationsfunktionen," *Z. angew. Math. Mech.*, v. 24, 1944, p. 73. 21.5×27.8 cm.

Tables with generally 9D are given for $n = 0(1)12$ of the quantities

$$\begin{aligned} (2n+1)J_n(\tau_k, \tau_{k+2}) &= P_{n+1}(\tau_{k+2}) - P_{n+1}(\tau_k) - P_{n-1}(\tau_{k+2}) + P_{n-1}(\tau_k) \\ (2n+1)K_{n,n}(\tau_k, \tau_{k+2}) &= 2\Delta\tau[P_{n+1}(\tau_{k+2}) - P_{n-1}(\tau_{k+2})] - J_{n+1}(\tau_k, \tau_{k+2}) + J_{n-1}(\tau_k, \tau_{k+2}) \\ (2n+1)L_{k,k+1,n}(\tau_k, \tau_{k+2}) &= 2(\Delta\tau)^2[P_{n+1}(\tau_{k+2}) - P_{n-1}(\tau_{k+2})] \\ &\quad - 4(\Delta\tau)[(2n+3)^{-1}\{P_{n+2}(\tau_{k+2}) - P_n(\tau_{k+2})\} - (2n-1)^{-1}\{P_n(\tau_{k+2}) - P_{n-2}(\tau_{k+2})\}] \\ &\quad + (\Delta\tau)[J_{n+1}(\tau_k, \tau_{k+2}) - J_{n-1}(\tau_k, \tau_{k+2})] \\ &\quad + 2(2n+3)^{-1}[J_{n+2}(\tau_k, \tau_{k+2}) - J_n(\tau_k, \tau_{k+2})] - 2(2n-1)^{-1}[J_n(\tau_k, \tau_{k+2}) - J_{n-2}(\tau_k, \tau_{k+2})] \end{aligned}$$

which arise from the integration of $P_n(\tau)$ multiplied by the first three terms in the Taylor series for $f(\tau)$. The intervals (τ_k, τ_{k+2}) are all of length 0.2, the range $-1, 1$ being divided at the points τ_k into equal parts of length 0.1. In the tables τ_k has the values $0(.2).8$. The coefficients J, K, L for negative values of τ_k are derived from those for positive values of τ_k by means of simple relations. With the aid of the tables the coefficients in the approxi-

mating Legendre series can be computed with the aid of a calculating machine. When the derivative of the function f is to be derived from the approximating series the value found by the author's method is generally satisfactory while that found by using Simpson's rule in the usual way, without any integration of Legendre functions in calculating the coefficients, is often incorrect.

H. B.

260[L].—GREAT BRITAIN, Admiralty Computing Service, Department of Scientific Research and Experiment, *Calculations involving Airy Integral for Complex Arguments—First [Second–Fifth] Zero*. 5 Offset prints of handwriting on one side of each of

- (i) 8 sheets (3 large folding, 39×32 cm.) dated Feb. 1944;
- (ii) 5 sheets (2 large folding) dated 21 Feb. 1944;
- (iii) 5 sheets (2 large folding) dated 21 Apr. 1944;
- (iv) 5 sheets (2 large folding) dated 21 July 1944;
- (v) 5 sheets (2 large folding) dated 28 Feb. 1945. These are respectively numbered SRE/ACS 21 [= NAO 6], 31, 39, 46, 55. 21.5×34.3 cm. These publications are available only to certain Government agencies and activities.

The functions tabulated are connected with the Airy Integral $Ai(z)$ which is defined by

$$Ai(z) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{1}{3}t^3 + zt\right) dt$$

and which satisfies the differential equation

$$\frac{d^3 Ai(z)}{dz^3} = z Ai(z), \text{ with } Ai z = 3^{-1/3}/(-\frac{1}{3})!, \text{ and } Ai'(z) = -3^{-1/3}/(-\frac{1}{3})! \text{ at } z = 0.$$

Tabulations are associated with each of the first five zeros of $Ai(z)$. For each zero Table I gives the values to 3D of $\alpha, \beta; \xi, \eta; -A, B$, for $t = 0(1)1$, and $10/k, k = 0(1)9$, where

$$\omega(t) = \alpha + i\beta = (\xi + i\eta)e^{iv/6} = -A + Be^{iv/3},$$

is that root of $Ai(z)/Ai'(z) = te^{5\pi i/12}$, which is equal to the specified zero of $Ai(z)$ at $t = 0$.

Table III gives, to 2D, the values of P and Q or λ and μ , where $P + iQ = e^{\lambda + i\mu} = F(t, x) = Ai(\omega + xe^{\pi i/3})/[e^{\pi i/3} Ai'(\omega)(1 - e^{\omega e^{-5\pi i/6}})^{1/2}]$, $t = 0(1)1$, $x = 0(1).5$ for P and Q , $x = 0(1)1(2)3$ for λ and μ . For x greater than 3, $\mu + \frac{1}{2}x^{2/3}$ is tabulated instead of μ ; an auxiliary table of $\frac{1}{2}x^{2/3}$ being given for $x = 3(2)50(5)100, 3-6, 3S; 6.2-28.2, 4S; 28.4-100, 5S. x = 3(1)10(10)100$ for λ and $\mu + \frac{1}{2}x^{2/3}$.

In order that values of $F(t, x)$ can be calculated for x greater than 100, Table II gives the values of γ and δ, t and $1/t = [0(1)1; 3D]$, where

$$e^{\gamma + i\delta} = \frac{1}{2}x^{-1/2}/[e^{5\pi i/12} Ai'(\omega)\{1 - e^{\omega e^{-5\pi i/6}}\}^{1/2}]$$

so that

$$e^{\lambda + i\mu} = F(t, x) = e^{\gamma + i\delta} Ai(\omega + xe^{\pi i/3}) 2x^{1/2} e^{\pi i/12}.$$

Extracts from introductory text.

261[L].—GREAT BRITAIN, Post Office Research Station, Central Radio Bureau, *Tables of the $J_n(x)$ Bessel Functions for Electrical Problems*. Investigation carried out by H. J. JOSEPHS, Miss M. SLACK, Miss S. M. GUMBREL, January, 1944. Research Report no. 12176, C.R.B. 44-986, OSRD Liaison Office no. W.A. 1720-7. 13 p. Offprint of typescript. 22×28 cm. Available only to certain Government agencies and activities.

Tables of $J_n(x)$, for $n = 0(1)10$, $x = [0(1)25(.5)50; 4D]$, have been assembled in this Report from various sources. They are required in the calculation of modulation products and certain transient oscillations. Parts of the $J_0(x)$ and $J_1(x)$ tables have already been published in the BAASMTTC, *Bessel Functions, Part I*, 1937, but the remainder of the tables, although included in the BAASMTTC program, is unlikely to be published for some years.

Extracts from the introductory text.

EDITORIAL NOTE: Much of these tables must have been "assembled" from manuscript sources, or computed, since tables for the range of values indicated have not been previously published. The BAASMTTC volume noted contains values of $J_n(x)$, $n = 0, 1$, for $x = [0(.001)16(.01)25; 10D]$; G. N. WATSON, *Theory of Bessel Functions*, 1922 and 1944, p. 730, has a table of $J_n(x)$, $n = 2(1)5$, $x = .1(.1)5; 7D]$, $n = 0(1)10$, $x = [1(1)12; 6D]$; JAHNKE & EMDE, *Tables of Functions*, 1938, 1940, 1943, 1945, has $J_n(x)$, $n = 0(1)10$, $x = [0(1)29; 4D]$, most of it abridged from a table by MEISSEL (*MTAC*, v. 1, p. 216, 298); K. HAYASHI, *Tafeln d. Besselschen . . . und anderer Funktionen*, 1930, has $n = 0(1)10$, $x = [.1(.1)5, 1, 2, 10(10)50, 100; \text{at least } 10D]$ (*MTAC*, v. 1, p. 216, 291). But even these far from suffice to give directly all the values of the table under review. The manuscripts BAASMTTC 2 and 6 (*MTAC*, v. 1, p. 283) would render all values through $x = 25.5$, but we know of no printed or manuscript table before 1944, giving $J_n(x)$, $n = 0(1)10$, $x = [26.5(.5)50; 4D]$, except the few items ($x = 27, 28, 29, 30, 40, 50$) indicated above.

262[L].—GREAT BRITAIN, H.M. Nautical Almanac Office, Department of Scientific Research and Development, *Tables of Incomplete Hankel Functions*. Photoprint on one side of each of 7 leaves; Offprint reproduction of handwriting and typescript tables. No. NAO 12, Nov., 1943. 32.2×20.5 cm. These tables are available only to certain Government agencies and activities.

The functions tabulated are functions of the two variables a and x , and are defined as follows:

$$(1) \quad A(a, x) - iB(a, x) = \int_0^x e^{2\pi i r} dr,$$

$$(2) \quad C(a, x) - iD(a, x) = \int_0^x e^{2\pi i r} dr/r,$$

where $r^2 = a^2 + x^2$.

The name *Incomplete Hankel Functions* is derived from the relation

$$(3) \quad H_0^{(2)}(2\pi a) = (2i/\pi) \{ C(a, \infty) - iD(a, \infty) \}.$$

The functions A, B, C, D have also been called *Generalized Exponential Integrals*, and are tabulated for the range $x = [0(.05)1.5; 4D]$, $a = 0(.1)5$. Although the last figure cannot be guaranteed it is unlikely to be in error by more than .7.

No attempt has been made to provide means for interpolation; if interpolation must be used, second differences will not suffice to yield four-decimal accuracy. Generally (i.e. except for C for $a = .1$ and small x) third and higher differences can be ignored, in interpolating in the x -direction, but must be included when interpolating in the a -direction.

Using (3)

$$C = -\frac{1}{2}Y_0(2\pi a) + P(a, x) \sin 2\pi x + (r/x)Q(a, x) \cos(2\pi x), \\ D = +\frac{1}{2}J_0(2\pi a) - P(a, x) \cos 2\pi x + (r/x)Q(a, x) \sin(2\pi x),$$

where

$$P \sim \frac{1}{2\pi x} - \left(\frac{1}{2\pi x}\right)' \left(2 + \frac{3a^2}{x^2}\right) + \left(\frac{1}{2\pi x}\right)'' \left(24 + 120 \frac{a^2}{x^2} + 105 \frac{a^4}{x^4}\right) - \dots,$$

$$Q \sim -\left(\frac{1}{2\pi x}\right)' + \left(\frac{1}{2\pi x}\right)'' \left(6 + \frac{15a^2}{x^2}\right) - \left(\frac{1}{2\pi x}\right)''' \left(120 + 840 \frac{a^2}{x^2} + 945 \frac{a^4}{x^4}\right) + \dots$$

$2\pi Q = dP/dx$. The check values at $x = 1.5$ for A and B are given by integrating the

equations

$$dA/d\alpha = -2\pi\alpha D, \quad dB/d\alpha = 2\pi\alpha C.$$

The values tabulated have been produced for a specific purpose; the notation used and the intervals and usages adopted would not necessarily have been adopted in other circumstances. The calculations are available to 6D for subsequent extension and interpolation, if required.

Extracts from introductory text.

263[L].—A. N. LOWAN and A. HILLMAN, "A short table of the first five zeros of the transcendental equation $J_0(x)Y_0(kx) - J_0(kx)Y_0(x) = 0$," *J. Math. Phys.*, v. 22, 1943, p. 208–209. 17.4×25.5 cm.

Table I of this paper gives values of the first 5 roots (not zeros!) x_s , $s = 1(1)5$, of the equation

$$(1) \quad J_0(x)Y_0(kx) - J_0(kx)Y_0(x) = 0$$

for $k = [1\frac{1}{2}(\frac{1}{2})4; 6D]$. If we substitute $kx = X$, $1/k = K$, the equation becomes

$$(2) \quad J_0(KX)Y_0(X) - J_0(X)Y_0(KX) = 0$$

which is of the same form. Thus Table I gives also the solutions to (1) when $1/k = 1\frac{1}{2}(\frac{1}{2})4$, although kx is tabulated in this case.

Table II gives values of $(k-1)x_s$, $s = 1(1)5$, for $k = [1\frac{1}{2}(\frac{1}{2})4; 6D]$. This table is more convenient for interpolation, but still does not suffice to give more than about 5S for intermediate values. The table could be extended backwards (i.e. for $k < 1$) by noting that

$$(k-1)x_s = -(1/k-1)kx_s = -(K-1)X_s,$$

and that $-X_s$ is also a root of (2). Interpolation by Newton's formula or Lagrange's formula might then be used, but turns out to give negligible gain in accuracy, owing to the singularity in the value of x_s for $k = 0$.

In order to produce a satisfactory table for interpolation it is clearly necessary to have a closer interval when k is near unity. It might also be worth while to explore the possibility of putting $k^2 = e^m$, as is done by DINNIK (see DINNIK 10 and 12, *MTAC*, v. 1, p. 222, 287) when giving solutions of

$$J_n(x)Y_1(xe^{1/m}) - Y_n(x)J_1(xe^{1/m}) = 0, \quad n = 0, 1.$$

Each zero is then an even function of m and advantage may be taken of symmetry properties.

Equation (1) is a particular case of the equation

$$J_n(x)Y_n(kx) - Y_n(x)J_n(kx) = 0$$

which arises when a cylinder function $aJ_n(x) + bY_n(x)$ is required, which shall vanish at two values of x that are in a given ratio $1:k$. The first published table giving roots of such equations seems to have been that of A. KALÄHNE, *Annalen d. Physik*, s. 4, v. 19, 1906, p. 88 (see *MTAC*, v. 1, p. 222, 294) and repeated, with additions, by him in *Z. Math. Phys.*, v. 54, 1906, p. 68, 81. These tables have been reproduced in the various editions of *Tables of Functions*, by JAHNKE & EMDE (p. 204–206 in the 1938 and subsequent editions). The values given in the latter are x_s and $(k-1)x_s$, $s = 1(1)6$, for $n = 0(\frac{1}{2})2\frac{1}{2}$, $k = [1.2, 1.5, 2; 4D]$ in each case, but with some uncertainty (indicated in the tables) in the last figure or two when $s = 1, 2$ or 3 for $k = 1.5$ or 2 . There is also a table giving the first root ($s = 1$) for $n = 0(\frac{1}{2})2\frac{1}{2}$, $k = [1, 1.2, 1.5, 2(1)11, 19, 39, \infty; 2-4D]$, and a table of $(k-1)x_s$, $s = 1(1)4$ for $n = 0, 1$, with 4 to 8 miscellaneous values of k to 3 or 4D.

Kalähne's table has never been completely superseded although the table under review is a considerable extension when $n = 0$. We may note that Kalähne's values of $(k-1)x_s$ for $n = 0$ are all correct to 4D, except that for $s = 1$, $k = 2$ the value can be improved to

read 3.1230. This item has not been altered on p. 206 of the 1945 edition of JAHNKE & EMDR's tables, although the value of x_n on p. 204 has been corrected.

A manuscript table by H. R. F. CARSTEN & Miss N. W. MCKERROW of British Insulated Cables, Ltd., Prescot, Lancs., England, gives x_n , $s = 1(1)6$ for $n = 0$ when $k = [2(4)6; 4D]$.

These tables have been compared with one another and have revealed only slight discrepancies which are evidently due to uncertainties of a unit or so in the final figure of the table by Carsten and McKerrow. No fuller investigation of accuracy has been made as it is evidently extremely improbable that any error remains to be found.

J. C. P. MILLER

264[L].—J. C. P. MILLER & C. W. JONES, "Notes on the evaluation of zeros and turning values of Bessel functions.—IV. A new expansion." *Phil. Mag.*, s. 7, v. 36, 1945, p. 200–206 + folding plate. 17×25.4 cm.

With the notation $xp = J_n(x)/J_{n-1}(x)$, $xq = J_n(x)/J_{n+1}(x)$, $1 + r = x_0/x$ where x_0 is a zero of $J_n(x)$, the aim is to expand r in series of powers of either p or q with coefficients which are functions of x . A partial differential equation for r as a function of p and x gives a recurrence relation for the coefficients a_n in the series

$$r = -a_1p - a_2p^2 - a_3p^3 - \dots$$

On p. 201 explicit expressions are given for the polynomials a_n for $n = 1(1)6$ and on a sheet facing p. 203 the polynomials with numerical coefficients are given for the ranges $n = -10(1) + 10$, $2n = -5(2) + 5$, $3n = -2(1) + 2$, $4n = -3(2) + 3$. The relation to the McMahon series is discussed and the initial approach to a zero is exemplified by the calculation of certain functions $B_n(x)$ defined by the Bessel-Lommel expansions. The zero of $Y_n(x)$, thus calculated, agrees with the accurate value to 11 significant figures, the error in the twelfth digit being less than one.

H. B.

265[L].—V. V. TARASSOV, "On a theory of low-temperature heat capacity of linear macromolecules," *Akademiâ Nauk, Moscow, Comptes Rendus (Doklady)*, v. 46, 1945, p. 22. 17.1×26.1 cm.

Two particular cases of a general formula are derived; the first leads to "the usual Debye equation for heat capacity"

$$c_1 = 36Ry^{-3} \int_0^y x^4 dx / (e^x - 1) - 9Ry / (e^y - 1);$$

and the second is given by the formula

$$c_1 = 6Ry \int_0^y x dx / (e^x - 1) - 3y / (e^y - 1), \quad y = \theta/T, \quad R = \text{gas constant}.$$

The values of c_1 were computed "with the aid of a very simple auxiliary method based on employing the table of Einstein functions for the heat capacity." There are tables of c_1 and c_2 , $y = [0(2)2(5)3(1)10(2)16, 20; 3D(9, 2D)]$. Compare *MTAC*, v. 1, p. 119, 189, 422.

266[L, M].—GREAT BRITAIN, H.M. Nautical Almanac Office, for the Admiralty Computing Service, Department of Scientific Research and Experiment, *Tables of the Function* $f(x) = e^{-x} \int_0^x e^{t/2} \cos t \sin^2 t dt$. Offset print on one side of each of the 6 sheets + "explanatory note," No. SRE/ACS 62, Nov., 1944. 21.5×34.3 cm. These tables are available only to certain Government agencies and activities.

The function $f(x)$ is tabulated for $x = [0(01)5(1)20; 5D]$, Δ . An auxiliary function, $x^{3/2}f(x)$, is also tabulated for $1/x = [0(01)2; 4D]$, Δ . Values of $f(x)$ are available to 7D

with an error not exceeding 2 units; the rounded-off values in the table have therefore a maximum error of .52. For interpolation second differences need not be used except between 5 and 7. The function was calculated from the formulae:

(a) $f(x) = \pi e^{-x}/(2x)[I_1(x) + L_{-1}(x) - 2/x]$ for $x = [0(.1)10; 7D]$, where the Bessel function $I_1(x)$ is given in B.A.A.S., *Mathematical Tables*, v. 6, and the Struve function $L_{-1}(x)$ in Applied Mathematics Panel, Report 59.1 (see SRE/ACS 38, item (g));

(b) $f(x) \sim (\pi/x)e^{-x}I_1(x) - e^{-x}(1/x - 1/x^3 - 3 \cdot 1^3/x^5 - 5 \cdot 3^3/x^7 - \dots)$ for the values $x = [10(.1)20; 7D]$. Values for $x = 0(.1)5$ were then interpolated to tenths on the National Machine, and the complete table made by rounding off to 5D.

The auxiliary function, $x^{3/2}f(x)$, was calculated to 7D from the formula:

$$x^{3/2}f(x) \sim x^{3/2}\{(\pi/x)e^{-x}I_1(x) - e^{-x}[1/x - 1/x^3 - 3 \cdot 1^3/x^5 - 5 \cdot 3^3/x^7 - \dots]\}$$

where $e^{-x}I_1(x)$ was taken from the B.A. Tables for $1/x \geq .05$ and calculated from its asymptotic expansion for $1/x < .05$.

Extracts from introductory text.

267[L, M].—GREAT BRITAIN, H.M. Nautical Almanac Office, for the Admiralty Computing Service, Department of Scientific Research and Experiment, *Tables of the Integrals*

$$A(x) = \int_0^x \frac{\cos(\pi t/2x)}{1+t^2} dt, \quad B(x) = \int_0^x \frac{\cos(\pi t/x)}{1+t^2} dt,$$

and

$$C(x) = \int_0^x \frac{\cosh(\pi t/x)}{1+t^2} dt.$$

Offset print on one side of each of the 6 sheets. No. SRE/ACS 22 = NAO 9, Dec., 1943. 21.5 × 34.3 cm. These tables are available only to certain Government agencies and activities.

In the expressions for $B(x)$ and $C(x)$, x is the smallest positive root of the equation $\tan z + \tanh z = 0$; to 8D, $x = 2.36502\ 037$. The three functions $A(x)$, $B(x)$, $C(x)$ are tabulated for $x = [0(.01)1; 4D]$, $1/x = [0(.01)1; 4D]$ each with Δ . The error in the last figure cannot exceed .53, with the exception of the following 37 values, in which the rounding-off adopted may be in error, the values given are correct.

$$\begin{aligned} A(x): x &= .52, .55; 1/x = .10, .16, .17, .45, .52, .57, .58, .71, .74, .76, .91, \\ B(x): x &= .33; 1/x = .09, .35, .41, .44, .46, .55, .64, .72, .77, .83, .89, .94, \\ C(x): x &= .04, .10, .42, .44, .47, .52, .74, .87, .93; 1/x = .32, .96. \end{aligned}$$

The integrals were calculated to 6D from the series expansions in terms of x and $1/x$ for the pivotal values, $x = -.2(.1) + 1.2$, $1/x = -.2(.1) + 1.2$. These values were then interpolated to tenths on the National Machine and rounded off to 4D. First differences are given. It will be seen that second differences rarely need to be used in interpolation, and can be ignored throughout, provided a maximum error of 1 unit in the fourth decimal can be tolerated.

The method of calculation, together with the differences of the pivotal values, suggests that the sixth decimal place of the pivotal values is correct to within 2 units; the method of interpolation used thus limits the accuracy of the interpolates to 3 in the sixth decimal. These values are available if required.

The series for $B(x)$ in ascending powers of x is

$$B(x) = B_1x - B_3x^3 + \dots + (-1)^n B_{2n+1}x^{2n+1} + \dots,$$

where

$$B_{2n+1} = (1/\pi)^{2n+1} \int_0^\pi \cos t dt,$$

and is calculated by means of a two-term recurrence relation. Similar series for $A(x)$ and $C(x)$ are obtained by the substitution of $\pi/2$ for z in the case of A , and of iz for z in the case of C .

The expansion in ascending powers of $1/x$ is $B(x) = \tan^{-1}x \cosh(x/z) - b_1(1/x) - b_2(1/x)^2 - \dots - b_{2n-1}(1/x)^{2n-1} - \dots$, where

$$b_1 = (x^2/2!) - (x^4/3 \cdot 4!) + (x^6/5 \cdot 6!) - \dots$$

$$b_2 = (x^4/4!) - (x^6/3 \cdot 6!) + (x^8/5 \cdot 8!) - \dots$$

$$b_3 = (x^6/6!) - (x^8/3 \cdot 8!) + (x^{10}/5 \cdot 10!) - \dots, \text{ etc.}$$

Similar series for $A(x)$ and $C(x)$ are obtainable by the substitutions given above.

Extracts from introductory text.

268[L, M].—GREAT BRITAIN, H.M. Nautical Almanac Office, for the Admiralty Computing Service, Department of Scientific Research and Experiment, *Tables of Certain Integrals*. Offset print of handwriting, and typescript tables, on one side of each of the 13 leaves. No. SRE/ACS 52, Sept., 1944. 21.5×34.3 cm. These tables are available only to certain Government agencies and activities.

The tabulated functions are

$$A_n(x) = \int_0^x \frac{\cos(2n-1)(\pi t/2x)}{1+t^2} dt, \quad B_n(x) = \int_0^x \frac{\cos(z_n t/x)}{1+t^2} dt,$$

$$C_n(x) = \int_0^x \frac{\cosh(z_n t/x)}{1+t^2} dt,$$

where z_n is the n th positive root of the equation $\tanh z + \tanh z = 0$, $n = 2, 3$ (the case $n = 1$ has already been tabulated in SRE/ACS 22). $z_2 = 5.49780392$, $z_3 = 8.63937983$.

The six functions are tabulated for $x = [0(.01)1; 4D]$, $1/x = [0(.01)1; 4D]$, with Δ . The error in the last figure is never greater than .54 and seldom exceeds .53. To preserve 4D accuracy in interpolation third differences can be ignored throughout, second differences must be taken into account for the functions $C_2(x)$, $C_3(x)$ for the whole range of x and $1/x$, but only in the range $0 \leq 1/x \leq .5$ for the remaining functions; elsewhere linear interpolation will suffice.

The integrals were calculated to 6D from the series expansions in terms of x and $1/x$ for the pivotal values

$$x = -.1(.05) + 1.1, \quad 1/x = -.1(.05) + 1.1,$$

except in the case of the functions A_2 , B_2 , A_3 , B_3 for which the chosen pivotal values were $x = -.2(.1) + 1.2$, $1/x = -.1(.05) + 1.1$. These values were then interpolated to fifths or tenths as necessary, and rounded off to 4D. The method of calculation and the differences of the pivotal values suggest that the error in the sixth decimal is less than 2 units, and the method of interpolation increases this error to something less than 4 units. These 6D values are available if required.

The series for $B_n(x)$ in ascending powers of x is very similar to the one already given for $B(x)$. For small x few terms are needed, but for x approaching unity the series converges very slowly, and direct summation becomes impracticable. Since the series $B_n(x)$ alternates in sign, after the first few terms, it can be transformed into a rapidly converging series by means of Euler's transformation. This states that if

$$\sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - \dots$$

is a converging series, then

$$\sum_{n=0}^{\infty} (-1)^n a_n = \sum_{n=0}^{m-1} (-1)^n a_n + \sum_{n=m}^{\infty} (1/2^{n+1}) \Delta^n a_n,$$

where $\Delta^n a_n$ is the n th ascending difference of a_n .

Extracts from introductory text.

269[M].—GREAT BRITAIN, Mathematical Group, The Malvern, Worcestershire, *Tables of the Incomplete Gamma Functions*, no. 1615/PMW, OSRD, Liaison reference no. WA 1464-8. August, 1945, 4 leaves, offprint typescript. 21.2×33.8 cm. Available only to certain Government agencies and activities.

$\Gamma(p) = \int_0^{\infty} e^{-x} x^{p-1} dx$. The incomplete Γ -function is defined to be $\int_0^y e^{-x} x^{p-1} dx$. The accompanying tables provide 4D values of the function

$$(1) \quad I = \Gamma(py, p)/\Gamma(p) = [1/\Gamma(p)] \int_0^{py} e^{-x} x^{p-1} dx,$$

where $p = m/(1 + 2m)$. Each value was originally computed to five figures from the series obtained by expanding the exponential and integrating. Being designed for a particular application, the tables are not otherwise comprehensive. The main tables are of I and $\log I$, for $\log y = [-5(.5) - 2(.1) + 1.5; 4D]$, $m = .1(.1)1, 2$.

Interpolation with respect to $\log y$ should be performed in $\log I$, and for this purpose central differences are provided, where necessary, for use with Besselian coefficients. Since the second differences, however, never exceed .003, linear interpolation will introduce an error of only .0004 in the worst case. Interpolation with respect to m is not intended.

For values of $\log y$ less than -5 , the formula

$$(3) \quad I \sim (py)^p/\Gamma(p+1)$$

suffices to give five-figure accuracy. For the use of (3) together with the function $N = mp^2 e^{-y}/\Gamma(p+1)$ a table is given of p , $\log [p^2/\Gamma(p+1)]$, and N , for $m = [.1(.1)1; 5D]$.

Extracts from introductory text.

EDITORIAL NOTE: The most elaborate *Tables of the Incomplete Γ -Function* edited by KARL PEARSON, were published at London in 1922. The greater part of the volume is taken up with the tabulation of $I(u, p) = \int_0^u e^{-x} x^{p-1} dx/\Gamma(p+1)$, where $z = (1+p)^{1/2}$.

270[M].—PETER L. TEA, "A graphical method for the numerical solution of Fredholm's integral equation of the second kind," *J. Math. Phys.*, v. 24, 1945, p. 114-120. 17.5×25.5 cm.

For the graphical conversion of a definite integral into a Stieltjes integral, when the integral is of trigonometrical type, graphs are given for the functions

$$S_q(y) = \int_0^y (1 + \sin qt) dt = y + (1/q)(1 - \cos qy)$$

$$C_q(y) = \int_0^y (1 + \cos qt) dt = y + (1/q) \sin qy$$

for $q = 1, 2$ and the range $y = 0$ to 2π . A chart of ordinate lines is given for $S_q(y)$ at intervals of 30° . The integral equation

$$F(x) = 1 + \frac{1}{2} \int_{-1}^1 e^{-|x-y|} F(y) dy$$

is solved graphically and compared with values obtained by BUCKLEY¹ and HEDEMAN.² A table with $x = [0(.25)1; 3D]$ is given for the functions $F_1(x)$, $F_2(x)$, ..., $F_5(x)$, obtained

by the method of successive substitutions. The kernel of the integral equation is an approximation to the actual kernel

$$k(y-x) = \frac{1}{2a} \left[1 - \frac{|y-x| \{ (y-x)^2 + 6a^2 \}}{\{ (y-x)^2 + 4a^2 \}^{3/2}} \right] \quad (a=1)$$

used by Hedeman* in his work with the cinema integrator. This kernel is quoted incorrectly in Tea's paper, the index 3/2 being omitted.

H. B.

* H. BUCKLEY, "Some problems of interreflection," Intern. Congress on Illumination, *Proc.*, 1928, p. 888.

* W. R. HEDEMAN, JR., "The cinema integrator in interreflection problems," *J. Math. Phys.*, v. 20, 1941, p. 415-416.

271[N, O].—J. F. STEFFENSEN, "A table of the function $G(x) = x/(1 - e^{-x})$ and its applications to problems in compound interest," *Skandinavisk Aktuarietidskrift*, v. 21, 1938, p. 60-71. 15 × 23 cm.

There is a table of $G(x)$, with Δ , for $x = [0(.01)10; 5D]$, and an auxiliary table of 100s, with Δ , for $200i = [0(.1)20; 5D]$. $e^i = 1 + i$, and, e.g., $a_m = n\delta/[iG(n\delta)]$. Since $G(x)$ requires frequent conversions of i to δ , and vice versa, the auxiliary table is useful, and its particular choice of interval planned so that linear interpolation should suffice in all cases.

Much earlier, in *Aktuaren, Nordisk Aktuarietidskrift*, Copenhagen, 1904, p. 50, N. P. BERTELSEN & J. F. STEFFENSEN gave a table of 100 $(e^x - 1)$, with Δ , for $100x = [0(.1)10; 5D]$. In *Institute of Actuaries, J.*, v. 15, part 6, 1870, p. 437-446, is "A table for determining the amounts, etc., of continuous annuities certain," by W. M. MAKEHAM, giving $\log [(e^x - 1)/x]$, with Δ and $\Delta' = \Delta^2/4$, $x = [0(.01)10.4; 7D]$.

In quite a different field we have already noted MTAC, v. 1, p. 119, "A six-place table of the Einstein functions," by J. SHERMAN & R. B. EWELL, $E = x/(e^x - 1)$, $-\ln(1 - e^{-x})$, $e^x E^2$, $x = [0(.005)3(.01)8(.05)15; 6D]$. See also RMT 198, v. 1, p. 422.

R. C. A.

272[U].—JUAN GARCÍA, [English t. p.:] *Astronomical Position Lines Tables. New and original tables, unique in saving of space and time, computed for the use with a new and original method of finding the observed position at sea and air. With titles and explanations both in English and Spanish. . . Volume II, Declination 25°N. to 25°S. Latitude 30° to 60°, N. and S.* [Spanish t. p.:] *Tablas de Líneas de Posición de Altura.* etc., Madrid, Editorial Naval, Montalbau 2, 1944. xviii, 127 p. 17 × 25.5 cm. 50 pesetas.

The tables in this volume were prepared for use with a new method, the involutes method, devised by the author, a lieutenant commander of the Spanish Navy, and explained in detail in his "Sobre el método de las curvas de altura envolventes," *Revista General de Marina*, 1943. The author tells us that this volume is the first of three to be issued; that v. I will cover declinations from 25°N to 25°S and latitudes from 30°N to 30°S; and that v. III is to cover the exact declinations of the navigational stars and latitudes from 60°N to 60°S.

The chief advantages claimed for the method are that the tables are compact and easy to use; a position line can be drawn rapidly enough for aerial navigation and with sufficient accuracy for surface navigation; altitudes near 90° can be used without hesitation; only one interpolation (for declination) is necessary and a convenient multiplication table, duplicated on a single loose sheet, is provided for that. Another great advantage not mentioned by the author is that the method is direct; one enters the table with the approximate observed altitude as one of three arguments and obtains an LHA or latitude. This is in

contrast to most navigation methods, where one enters with an estimated LHA and obtains a computed altitude for comparison with the observed altitude.

The tables are divided into two parts; the first 72 pages cover the cases where the latitude and declination are of the same name and 53 pages take care of "opposite name." One or more double-entry tables are given for each integral value (in degrees) of altitude, 5° to 83° for "same name," 5° to 59° for "opposite name." The horizontal argument in each table is declination, in integral degrees 0° to 25°. For altitudes of bodies near the prime vertical, the vertical argument is latitude, 30° to 60°, with interval of argument varying from 1° to 3°, and the tabulated quantities are meridian angle to the nearest minute (printed without the usual symbols for degrees and minutes and without a space between the numbers of degrees and minutes) and the change in meridian angle to the nearest minute corresponding to 10' change in declination, printed in red. For altitudes of bodies near the meridian, the vertical argument is local hour angle, always an integral number of degrees but with varying interval of the argument, and the tabulated quantities are latitude to the nearest minute (printed as indicated above for local hour angle) and the change in latitude corresponding to a change of 10' in declination, printed in red.

Although nominally the tabulated values are given to the nearest minute, the use of a simple device allows one to determine values to the nearest half-minute. Values in the table with the terminal digit underlined are to be put down as 0.3 less; those without a line under the terminal digit are to be set down as 0.2 greater. Thus 1624 means 16° 23.7'; 1426 means 14° 26.2'.

Only a single example is worked out, one that has been used in *Admiralty Navigation Manual*, v. 2, 1938, (a), p. 136f, by the haversine method; (b), p. 157f, by Hughes' *Tables for Sea and Air Navigation*; and (c), p. 163, by *Tables of Computed Altitude-Asimuth* (H.O. 214). This lone example shows how a table with vertical argument latitude is used; probably many users would have appreciated a second example showing the use of a table with vertical argument local hour angle.

Throughout the table are columns of red figures giving the change in tabulated values due to a change of 10' in declination. Underlined end-figures are also here in evidence.

The involutes method requires the user to enter the table with the corrected altitude and the declination, each rounded off to the nearest degree, and two adjacent latitudes (or local hour angles), and to take out the corresponding local hour angles (or latitudes). These will give two points which will serve as centers of circles, the radii of which are taken as the difference between the corrected altitude and the integral value used as an argument in entering the tables. The line of position is taken as one of the external tangents to the two circles thus obtained, the choice being made by the same rules that enable one to decide in the usual methods whether to move toward or away from the observed celestial body.

The method does not provide a computed azimuth of the celestial body directly, but one can read that from the chart on which the line of position is plotted. The method requires a chart covering several degrees in both latitude and longitude.

A comparison of a hundred values from the table, fifty in the "same name" portion, fifty in the "opposite name," with corresponding values determined from HO 214, v. 4, showed eight values in a hundred in error by more than 0.3, six of 0.4 and two of 0.5.

At the end of the preface appears the following paragraph: "It is the author's belief that no other book in the world brings to the problem the simplicity and accuracy in compact tables that this one does, it being therefore unique in the saving of space and time. Hence the author hopes that these tables will be useful to navigators all the world over." At first reading, the user will suspect that the author may have exaggerated somewhat the value of his tables; after using them, he will be less inclined to feel that the author has overrated them. Garcia deserves high praise for his valuable contribution to the science of celestial navigation; his method and his tables will undoubtedly be widely used in the years to come.

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273[U].—HYDROGRAPHIC OFFICE, *Publication no. 351, v. 4, Secret, Celestial Air Navigation Table. Volume 4, Latitudes 30°–40°*, Tokyo, May, 1941. iii, 340 p. 18 × 25 cm. While this publication is in Japanese the star names are in English, although there is a sort of equivalent auxiliary Japanese interpretive system. Arabic numerals and the ordinary signs for degrees and minutes are also used, and Japanese equivalents are indicated in a list of English terms used in celestial navigation. KOIKE SHIRO signs the preface as director of the Hydrographic Office.

This is volume four and the first to be issued of a planned series of six volumes of air navigation tables published by the Japanese Government. Each volume is to cover eleven degrees of latitude, with a degree of overlap between consecutive volumes. A copy of this volume was sent to the Library of the U. S. Hydrographic Office before Pearl Harbor, in the normal exchange of such material; it was very courteously made available to the reviewer, by G. A. PATTERSON, commander in the U. S. Navy.

For each integral degree of latitude, 30°–40° inclusive, each integral degree of local hour angle, and each degree of declination, -29° , $+29^\circ$ inclusive, there are tabulated the altitude to the nearest minute (and down to $2''$), the azimuth angle to the nearest degree and a correction factor to be used in allowing for minutes of declination. For a selected list of 28 bright navigation stars with declinations lying outside the range, -29° , $+29^\circ$, there are tabulated for each integral degree of latitude, 30°–40° inclusive, and for each integral degree of local hour angle, the altitude to the nearest minute (and down to $4''$) and the azimuth angle to the nearest degree. All altitudes are corrected for refraction at an elevation of 4000 meters. A table is provided for correcting for refraction at other altitudes.

All of the data corresponding to a single latitude are gathered together in a single division with the declination section preceding the star section. This is convenient for use, though occidental readers may have some difficulty with the arrangement which finds data for both North and South latitudes on the same page, with the North material reading down and the South material reading up.

From the declinations of stars tabulated in the front, and from the computed altitudes for stars of high declination, it is found that the epoch of the star positions used is 1945, several years *after* the date of issue. This represents good practice since it allows a table to be used longer before it becomes obsolete.

From internal evidence, one can judge that refraction to the nearest tenth of a minute was added to an altitude computed to the nearest tenth of a minute, and the result rounded off to the nearest minute to provide a tabulated altitude. One might suspect at first sight that the tables were copied from the British Astronomical Navigation Tables, but the Japanese tables give altitudes for Sun, Moon and Planets down to $2''$, and for stars down to $4''$, while the ANT omit all altitudes greater than $80''$, or less than $10''$. Also, after allowing for differences in refraction due to the difference in elevations for which the two sets of tables were computed, the tabular values do not agree. There is some evidence that the Japanese used H.O. 214 in preparing part of their tables, but the small number of errata in the American tables makes it difficult to prove.

Auxiliary tables included in the volume are one for changing hours, minutes and seconds into degrees, minutes and seconds; one for parallax of the moon; and one for dip of the horizon.

CHARLES H. SMILEY

EDITORIAL NOTE: In *Log of Navigation*, v. 3, March, 1945, p. 12–14, is an article entitled, "A mystery and a clue," by E. A. Snyder and B. H. Bogdikian. On 1 Feb., 1945, 18 charred page-fragments of an air navigation table had been found in a plane destroyed on Saipan; portions of four of the pages were reproduced in the article. Their "mystery" was cleared up when it was shown (*idem*, July, 1945, p. 28) that the pages were part of the volume reviewed above.

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274[U].—JOOST H. KIEWIET DE JONGE, *Three-Star Position Tables for Aerial Navigation. Latitudes 20° to 30° North*. New York, Hydrographic Office, U. S. Navy, New York Project, 1945 [94 p.], 24 × 33 cm. Not available for public distribution. Printed throughout by the offset process.

This single volume of aerial navigation tables, by a lieutenant of the air force in the Netherlands East Indies Army, was issued by the Hydrographic Office of the U. S. Navy to determine by actual use the practical value of this particular type of table. It allows the determination of an observer's longitude and latitude directly from the altitudes of three selected stars, measured in a particular order at three-minute intervals. Refraction, presumably corresponding to an elevation of 8,000 feet above sea level (though not so specifically stated), is included in the altitudes, h_1, h_2, h_3 (each to the nearest minute) which are used as entering arguments in Table I. This table gives $H_1 = \log \sin h'_1, H_2 = \log \sin h'_2, H_3 = \log \csc h'_2$ where h'_1, h'_2, h'_3 are the altitudes corrected for refraction. The sums, $H_{12} = H_1 + H_2, H_{23} = H_2 + H_3$, are used as arguments in entering the main tables which are double-entry tables, one for each set of three stars. Tabular differences are given and one obtains directly latitude and equinoctial longitude, and then the terrestrial longitude by adding the Greenwich hour angle of the vernal equinox, interpolated from the Air Almanac for the time of the second observation, to the equinoctial longitude. Table II allows one to take into account the observer's displacement in flight, and Table III permits one to correct for intervals between successive observations of more than three minutes.

Only twenty stars appear in the tables, in fourteen combinations of three each; it is pointed out in the introduction that this is considered a minimum set. The altitude of each star of a group of three must lie between 20° and 75°, and the difference in azimuth between successive stars must not be greater than 165°. Also, as mentioned above, their altitudes must be measured at three-minute intervals in a particular order. The limitations on the altitudes just mentioned mean that only 62% of the visible celestial hemisphere can be used at any time with these tables, as against 81% with HO 218, or 91% with HO 214.

Refraction has been allowed according to the critical table:

h	20°	25°	60°	75°
R		- 2'	- 1'	0'

For an elevation of 8000 feet above sea level, it would have been better if the 25° and 60° had been replaced by 27° and 55° respectively.

The theory of the tables is not given in the volume, but it is easily reconstructed. If we note that

$$\begin{aligned} H_{12} &= \log (\sin h'_1 / \sin h'_2) = \log C_1 \\ H_{23} &= \log (\sin h'_2 / \sin h'_3) = \log C_2 \end{aligned}$$

where C_1 and C_2 may be considered as known from observation, one sees that the local hour angle of the second star, t_2 , at the time of its observation, may be found from the equation:

$$\begin{aligned} (C_2 \sin d_2 - \sin d_3) [\cos d_1 \cos (t_2 + c_1) - C_1 \cos d_2 \cos t_2] \\ = (C_1 \sin d_2 - \sin d_1) [\cos d_3 \cos (t_2 + c_3) - C_3 \cos d_2 \cos t_2]. \end{aligned}$$

And the latitude, L , corresponding to the time of observation of the second star, may be found from the equation:

$$\tan L = [\cos d_1 \cos (t_2 + c_1) - C_1 \cos d_2 \cos t_2] / [C_1 \sin d_2 - \sin d_1].$$

In these equations, d_1, d_2, d_3 are the declinations of the three stars and $(t_2 + c_1)$ and $(t_2 + c_3)$ are the local hour angles of the first and third stars at the times of their observation. Thus c_1 and c_3 are known.

It is suggested that the time required for the reduction of three-star observations is reduced by a factor of four; to the reviewer, this factor seems high. Also it is to be noted that the Jonge solution gives no hint as to the azimuths of the three stars used, and hence

no approximate check on the course of the plane or the compass used. Nor is any clue given as to which observation may be in error in case of a discrepancy. The method provides no alternate procedure in the event only two of the three stars are observed. The old-fashioned method of plotting lines of positions may be slow and somewhat tedious, but in time, most navigators learn to interpret the lines to good advantage. The lack of similarity to earlier methods of celestial navigation is a definite disadvantage; the navigator examining Jonge's method for the first time will probably feel that he must learn an entirely new technique.

CHARLES H. SMILEY

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT 233 (Fletcher, Miller & Rosenhead), 240 (Bretschneider), 243 (Jahnke & Emde, Meissel, Smith), 245 (Gupta), 247 (D. N. Lehmer), 250 (Vallarta), 255 (H.B. & R.C.A.), 263 (Kalähne), 270 (Tea), 274 (Jonge); N 50 (Fadle).

70. EDWIN P. ADAMS, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, first reprint, Washington, 1939. Compare *MTAC*, v. 1, p. 191, 325.

A. P. 127, formula 6.475, no. 2, for $(2^4/2!)x^3 - (2^4/6!)x^4 + (2^4/10!)x^{10}$,
read $(2/2!)x^3 - (2^3/6!)x^4 + (2^3/10!)x^{10}$.

P. 136, line 2, the lower limit of the integral is 0.

P. 197, formula 9.110, no. 2, for $C_{-1}(x)$, read $C_{-1}(x)$; no. 4, for $dC(x)/dx$, read $dC_r(x)/dx$.

P. 199, formula 9.151, for $2(\frac{1}{2}x)$, read $2(\frac{1}{2}x)^r$, and in the integral, for the upper limit π , read $\frac{1}{2}\pi$.

P. 203, formula 9.202, line 3, for $J(x)$, read $J_{3/2}(x)$.

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B. P. 189, formula 8.708, no. 1, for c_1x , read c_1x^3 ; formula 8.709, for $(a + b^2x^2)y$, read $(a + b^2x^2)y$

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EDITORIAL NOTE: The error noted on p. 127 of *A* was copied by L. B. W. JOLLEY, in formula (508) of his *Summation of Series*, London, 1925.

71. H. BRANDENBURG: 1. *Siebenstellige trigonometrische Tafel* . . . , second ed., Leipzig, 1931, p. 336.

2. *Sechsstellige trigonometrische Tafel* . . . , Ann Arbor, Mich., 1944, p. 300; compare *MTAC*, v. 1, p. 387 f.

The 30-place tables in question are those of (a) $(\pi/2)^n$; and (b) $(\pi/2)^n/n!$

$(\pi/2)^n$

n	For	Read	n	For	Read
7	...27019	...27125	14	...15909	...19020
8	...62080	...62136	15	...37037	...40617
9	...64804	...64921	16	...39780	...43355
10	...58511	...58710	17	...12705	...23698
11	...98520	...98821	18	...18723	...31773
12	...62827	...63300	19	...43830	...72517
13	...07103	...07788	20	...367066	...400842

The existence of errors in the 1932 edition, of the second work listed above, was referred to by L. J. C. in *MTAC*, v. 1, p. 162, but he gave no details as to their identity.

$$(\pi/2)^n/n!$$

<i>n</i>	<i>For</i>	<i>Read</i>
10	...53002	...53022
13	...667 92681 17753	...967 92681 17753.

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72. BURROUGHS ADDING MACHINE Co., *Table of Reciprocals 1 to 10,000*, Detroit, Michigan, 1911, 24 p. Revised ed., 1940. With a thumb index for each 1000. 22.2×27.7 cm.

In 1930 this table was compared with proofs of Barlow's *Tables* (*MTAC*, v. 1, p. 16-17); the only errors found were:

Reciprocal of 118, *for* 8474 577, *read* 8474 576;
185, *for* 5405 406, *read* 5405 405;
476, *for* 2100 841, *read* 2100 840.

In editions of that date the argument 8260 was printed 6260, but in the revised edition it has been corrected. The purpose of this note is not so much to call attention to these trivial errors, none of which would affect a calculation, as to point out the general correctness of the table.

This table was produced for use with the Burroughs Moon-Hopkins multiplying machine and with their key-driven calculator, in order that division might be done by multiplying by a reciprocal. It is excellent for this purpose.

L. J. C.

73. P. R. E. JAHNKE & F. EMDE, *Tables of Functions with Formulae and Curves*, fourth ed., 1945. Compare *MTAC*, v. 1, p. 386, 391 f.

A. Besides 17 last-place unit errors I found the following other errors in the second table of powers in the Addenda, p. 8, 9:

Argument	Power	<i>For</i>	<i>Read</i>
.7	.15	.9497	.9479
.8	.15	.9673	.9671
.8	.20	.9551	.9564

An additional error in this table was reported by J. C. P. MILLER, *MTAC*, v. 1, p. 397.

JOHN W. WRENCH, JR.

EDITORIAL NOTE: The errors occur also in EMDE, *Tables of Elementary Functions*, 1944, p. 8; see *MTAC*, v. 1, p. 384.

B. P. 274, *for* F. C. Titchmarsh, *read* E. C. Titchmarsh; error since the 1933 edition.
P. 275, *for* $x^{-7/12}e^{-x/12}M$, *read* $x^{-7/12}e^{-x/12}M$; error since the 1938 edition.

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74. E. MEISSEL, "Abgekürzte Tafel der Bessel'schen Functionen $I_n^{(a)}$," *Astr. Nach.*, v. 128, 1891, col. 156. Compare *MTAC*, v. 1, p. 298.

In this table are the following errors:

$J_{100}(1000)$, for +.0075 2995, read +.0075 3142,

$J_{100}(1000)$, for -.0052 6524, read -.0052 6276.

My results were obtained by computing the two values $J_{1000}(1000) = .0447\ 3067\ 295$, $J_{100}(1000) = .0488\ 3022\ 878$, from formulas given by Meissel, and an application of the recurrence formula. For $n = 981(1)1000$, I checked his results exactly.

M. S. CORRINGTON

75. NYMTP, *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, Washington, 1939. See *MTAC*, v. 1, p. 45 f, 161.

On p. 337, in the argument following the argument 1.6847, for 1.6868, read 1.6848.

A. N. LOWAN

76. D. B. SMITH, L. M. RODGERS & E. H. TRAUB, "Zeros of Bessel functions," *Franklin Institute, J.*, v. 237, 1944, p. 301-303. Compare *MTAC*, v. 1, p. 213, 215, 217, 218, 274, 305. On this last page we quoted the authors' statement on p. 303, that these tables are believed to be "accurate to the extent that the last figure is within plus or minus two of the correct value." On 4 June 1945, Mr. M. S. CORRINGTON (see *MTAC*, v. 1, p. 212, 285 and *RMT* 222) drew our attention to serious discrepancies between results of these authors and those of G. N. WATSON and J. R. AIREY. Resulting investigations led to the following report:

The whole of Table II has been compared with BAASMTTC results, obtained for $x < 25$ by Mr. S. JOHNSTON, see *MTAC*, v. 1, p. 284, BAASMTTC 9. This has confirmed the three major errors found by Corrington:

$j_{2,7}$ for 24.27112, read 24.27011,

$j_{4,8}$ for 24.1990, read 24.0190,

$j_{4,1}$ for 8.77142, read 8.77148.

Three other changes were also indicated, but in each case the value given is within a unit of the true value. Other values are correct; in particular $j_{2,4}$ and $j_{2,6}$ are correct, Watson's values being in error (see *MTE* 78).

For Table I, no BAASMTTC results are yet available, except for $n = 0$ (which gives the same zeros as $n = 1$ in Table II), but for $n \leq 4$ a comparison was made with another table available to the writer. For $n > 4$, the columns of values were differenced, together with the values for $n \leq 4$; this provided a complete check on the run of the values, as the differences are well-behaved, especially so for high n . The final digit is not completely checked in this way but, apart from three errors listed below, it is certainly within the limit claimed by the authors, and almost certainly within one final unit, i.e. of the fourth decimal, of the true value. Finally the last value in each column for $s = 1(1)7$ (the last 4 values for $s = 5$) were recomputed from the equation $J_{n-1}(x) = J_{n+1}(x)$, using the BAASMTTC 10-decimal table of $J_n(x)$ at interval 0.1 in x (see *MTAC*, v. 1, p. 283, BAASMTTC 2). This was done either by inverse interpolation, or by use of the approximate formula

$$j'_{n,s} = x + \frac{x^3}{x^2 - n^2} u - \frac{x^3(x^2 + n^2)}{2(x^2 - n^2)^2} u^2 + \dots$$

in which x is an approximation (here a multiple of 0.1) to the zero, and $u = J'_n(x)/J_n(x)$.

Three major errors were found:

$j_{k,1}$ for 22.6721, read 22.6716,
 $j_{k,1}$ for 24.1469, read 24.1449,
 $j_{k,1}$ for 23.8033, read 23.8036.

Five other changes (there may be more) were also indicated, but each value given is already within a unit of the true value.

In two tables giving 164 values in all (only 156 distinct and non-trivial) there are thus 6 major errors. This is too high a proportion and, in the present age of calculating machines and of the production of large books of mathematical tables with a good chance of being error-free, it seems to the writer that it is time both that (a) authors of small tables should make *certain* that the tables are accurate (a table is not easy to compute unless it is easy to check by an independent method), and that (b) editors should do their utmost to choose a referee who is willing to check the tabular material, especially if the table is of general importance.

J. C. P. MILLER

2 August 1945.

EDITORIAL NOTE: On Aug. 13, 1945, Mr. M. S. Corrigton also drew our attention to the fact that the tables of Smith, Rodgers & Traub were reprinted in *Electronics*, v. 17, July 1944, p. 240, 244, 248. Hence the above criticisms apply equally to this reprint.

77. U. S. Hydrographic Office, *Useful Tables. From the American Practical Navigator*, H.O. 9, part II, Washington, D. C., 1911. Tables of Logarithmic and Natural Haversines, p. 817-921.

As a result of an entire recomputation of the Haversine tables taken from this work (described *MTAC*, v. 1, p. 421-422), I have to report that the above Hydrographic Office table contains at least 230 unit errors in the last decimal place and the following more serious errors:

Page	Angle	For	Read
818	2° 52'	6.79630	6.79636
821	11° 09'	7.97478	7.97487

CHARLES A. HUTCHINSON

University of Colorado
 Boulder

78. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*. Cambridge, University Press, 1922. Second ed., Cambridge, University Press, and New York, Macmillan, 1944. See *MTAC*, v. 1, p. 296, 307, 364, 367. The editors received from H. W. SANDERS of the Univ. Adelaide a list of 50 errors in the text of the first edition, but not affecting tabular material, copied from the late J. R. Wilton's copy of this work.

The following list of errata is the result of a fairly systematic, but incomplete, examination of the numerical tables by comparison with other tables, published and unpublished. The unpublished tables are mainly ms. tables prepared for future BAASMTTC volumes. It would take a long time to complete the checking of all the tables in Watson's book, so that it seems worth while to give an interim list. It also seems worth while, on the other hand, to include *all* major errors, even if they have been noted previously in this journal. Besides errors in the numerical tables, the list includes also some errors in the text that are known to the writer, and two of the corrigenda given by Watson himself in the first edition, but often overlooked.

Page	Line or function	For	Read	Authority or reference
62	11	$\sum_{m=0}^{\infty} \frac{(n-m-1)!}{m!} (\frac{1}{2}x)^m$	$\sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} (\frac{1}{2}x)^m$	Watson, 1st ed. only
117	6 up	ux^{-}	ux^{-1}	J. R. Wilton, corrected in 2nd ed.
228	4, Denom.	3072	192	Lehmer,
	Num.	768	48	MTAC, v. 1,
		41280	2580	p. 134
	8, Num.	98720	78720	Watson, 1st ed. only
374	(4)	$\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + 1)$	$2\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + 1)$	Bateman
416	Formula (3)	$-\frac{1}{2}\pi(\nu - \mu - \frac{1}{2})$	$-\pi(\nu - \mu - \frac{1}{2})$	E. T. Goodwin
563	Last, Num.	1	$(2m)!$	A. Erdélyi
564	3, Num.	1	$(2m-1)!$	"
655	5, 6			See note (1)
	10 up	0.2	0.1	Miller
661	9	9.0	8.9	"
	14	11.0	15.0	"
664	10 up	50 ... 1	1 ... 0.02	"
671	$Y_1(2.96)$	0.3136280	0.3136281	See note (2)
698	$e^K K_0(x), x = 0.14$	2.4122685	2.4123173	"
	$e^K K_1(x), x = 0.14$	8.0076828	8.0076794	
726	$J_{1/2}(12.82)$	0.1087869	0.1087689	Miller
730	$J_4(3.6)$	0.2197990	0.2197991	See note (3)
	$J_4(3.6)$	0.0896796	0.0896797	"
732	$Y_{12}(7)$	-37.8317507	-37.8317508	"
	$Y_{12}(8)$	-9.5431018	-9.5431019	"
734	$Y_7(1.9)$	-382.366	-382.3664	"
739	$K_3(1.6)$	13717.316	13717.317	"
740 to				
743	$J_{-2n}(11)$	-0.066647	-0.066648	See note (4)
	$J_{-2n}(16)$	-0.123323	-0.123332	
	$J_{12n}(11)$	-0.101814	-0.101815	
	$J_{-12n}(16)$	-0.010308	-0.010299	
	$J_{12n}(11)$	+0.133432	+0.133431	
	$J_{22n}(11)$	1086	1089	
	$J_{22n}(11)$	355	356	
	$J_{22n}(12)$	3532	3533	
744	$x = 1.5$	0.415348	0.415483	See note (5)
	$x = 11.0$	0.380390	0.380392	
		0.504784	0.504786	
	$x = 11.5$	0.395149	0.395152	
	$x = 47.5$	0.479313	0.479311	
748 to				
750	$y_{1,n}$	115.4503820	115.4502820	See note (6)
	$j_{1,1}$	16.2234640	16.2234662	
	$j_{1,2}$	19.4094148	19.4094152	
	$y_{1,1}$	14.6230726	14.6230777	
	$y_{1,2}$	17.8184543	17.8184552	
	$y_{1,3}$	20.9972845	20.9972848	
	$j_{1,1}$	7.5883427	7.5883424	
751	2 up from table	Delete the second $\sqrt{3}$		Miller
	s_2	5.6101956	5.5101956	R. P. Bell,
				Miller
777	Otti	61-96	1-56	Miller

NOTES

- (1) Confusion here; both Hansen and Schlömilch give the table for $\frac{1}{2}\pi = 0(.05)10$, Lommel's table is for $\pi = 0(.1)20$, and is not an extension. See *MTAC*, v. 1, p. 194, MTE 29.
- (2) These errors were noted in BAASMTc, *Bessel Functions, Part I*, p. xvii. All are corrected in the second edition, the correction to $Y_1(2.96)$ being trivial.
- (3) All of Table IV has been compared with BAASMTc ms. tables, with the exception of the tables of $e^{-x}I_n(x)$ on page 736. All errors found are listed.
- (4) The table of $J_{n(n+1)}(x)$ for $x \leq 20$ was compared with Airey's table in the B.A.A.S., *Report for 1925*; for $x > 20$, the table is unchecked. All discrepancies of a unit or more in the sixth decimal have been listed; there are also 42 others less than a unit in amount. The error in $J_{-22}(16)$ was noted by Airey.
- (5) Given by J. W. Wrench, Jr., *MTAC*, v. 1, p. 366, MTE 58; they were known to Airey for $x \leq 20$. Wrench also gives 26 further changes of a unit in the sixth decimal.
- (6) These discrepancies were found by comparison with BAASMTc ms. tables, by S. Johnston and the writer. Eight other discrepancies of a unit were also noted. The ms. tables have not yet been fully checked, so that this list must be regarded as provisional.

J. C. P. MILLER

UNPUBLISHED MATHEMATICAL TABLES

Reference has been made to an unpublished table in RMT 262 (Great Britain), 263 (Carsten & McKerrow), 266 (Great Britain), 267 (Great Britain), QR 20 (Miller & Johnston).

39[B].—*Table of Powers of $z = x + iy$* . Manuscript prepared by, and in possession of, the NYMTP.

This ms. gives the exact values of z^n for $n = 1(1)25$, $z = x + iy$, where x and y each ranges from 0 through 10 at unit intervals.

A. N. LOWAN

40[L].—ENZO CAMBI, *Tables of $J_n(x)$* . Manuscript in the possession of the author, a doctor of engineering, Via Giovanni Antonelli 3, Rome, Italy.

These are tables that I have calculated in recent years. The first contains $J_n(x)$ for $x = [0(.001).5; 15D]$ and $n = 1(1)11$, that is to say, to that value of n where $J_n(0.5)$ is of the order of 10^{-18} . Such a table, in conjunction with the well-known addition formula for the Bessel functions, and, for instance, with Meissel's table of $J_n(x)$ for integral values of x up to 24 (see *MTAC*, v. 1, p. 216), makes it possible to calculate directly $J_n(x)$ for $x = [0(.001)24.5; 15D]$.

The second table gives $J_n(x)$ for $x = [0(.01)10.5; 10D]$ and $n = 1(1)29$. By the use of this table it is possible to compute easily the value of any function given in the form of a Neumann series of Bessel functions for $x = [0(.01)10.5; 9 \text{ or } 10D]$. This range of x covers the field of most applications in physics.

For the main table, values of $J_n(x)$ for $x = 0(.05)10.5$ were first computed to 12D, for even n up to $x = 1.5$, power series being used; for higher values of x , derivatives at unit interval of x formed from Meissel's table were used to get values at interval .05 with the aid of Taylor's series. These were checked by differencing, subtabulated to interval .01, and the final values checked by

$$J_0 + 2J_2 + 2J_4 + 2J_6 + \dots = 1.$$

Values for odd n were formed by recurrence and checked by

$$J_1 + 3J_3 + 5J_5 + 7J_7 + \dots = \frac{1}{2}\pi.$$

ENZO CAMBI

EDITORIAL NOTE: A great deal of these tables overlap the still unpublished tables of $J_n(x)$ that L. J. C. prepared for the British Association in 1935 and 1936, covering values of x up to 25 and n up to 20, part with 8D and part with 10D. See *MTAC*, v. 1, p. 283.

41[L].—A. OSWALD ALWIN WALTHER (1898–), who directed the preparation during 1944–45 of 9 Reports on *Tables of Bessel Functions*, at the Institute of Practical Mathematics, Technische Hochschule, Darmstadt, Germany. B. BAASMTTC, Unpublished *Tables of Bessel Functions*. In some respects the summary differs from that given *MTAC*, v. 1, p. 282–284. Descriptions of A–B published in Great Britain, Admiralty Computing Service, Department of Scientific Research and Experiment, *Miscellaneous Information Sheet—III*, no. SRE/ACS 91, August, 1945, 5p. Mimeographed. 21.5 × 33 cm. Since this *Information Sheet* is available only to certain Government agencies and activities, the British Admiralty has given us permission to reprint the greater part of it in what now follows. It was O. A. Walther, and others, who computed the table of $\Lambda_n(x)$ in JAHNKE & EMDE, *Tables of Functions*, 1933+.

1. Introduction.

Certain unpublished tables of Bessel functions are now in the possession of Admiralty Computing Service. Though general circulation of these tables is impossible it is desirable that the values should be made available to any establishment having a requirement for them. The object of this report is to inform establishments what requests for values of Bessel functions can be met immediately by Admiralty Computing Service. Though, as a rule, particular values only can be supplied, in a few cases a complete table is available on loan, this fact being indicated in the text of this report.

2. List of unpublished German tables.

Report No. 1.

$J_n(x)$ to 4S for $n = 9(1)15$, $x = 4.5, 4.8, 5.2, 5.5, 6.5, 7.2, 7.5$.

Report No. 2.

Zeros j'_{nm} of J'_n , together with values of $J_n(j'_{nm})$ to 7D for $n = 1(1)4$, $m = 1(1)10$. Also for $n = 1$, $m = 11$.

Reports Nos. 3, 5 and 8.

These three reports contain $J_n(x)$, for $n = 0(1)30$, $Y_0(x)$ and $Y_1(x)$ to 7D for $x = 0(.2)65$. For x less than n , values less than 10^{-7} are not tabulated, while values between 10^{-7} and 10^{-2} are given to 6S.

Report No. 4.

The coefficients $(n + \frac{1}{2}, m)/2^n$ of Hankel's asymptotic series for $J_{n+\frac{1}{2}}(x)$. These are tabulated to 8S for $n = 0(1)30$, $m = 0(1)n$.

Reports Nos. 6, 9.

These two reports contain $J_{n+\frac{1}{2}}(x)$ for $n = -31(1) + 31$, $x = 0(.2)20$; 5S being given for negative values of n , 5D elsewhere.

Report No. 7.

Some numerical examples on the calculation of Bessel functions.

Particular values of the above quantities, or of any deducible from them, can be supplied when required.

3. General review of the German tables.

All the above reports were reproduced by a simple bromide process, with the result that the legibility varies considerably from page to page, according to the quality of the

printing: in general the figures are clear but some of the badly printed pages are almost illegible.

A rough indication of the number of errors to be expected was obtained by examination and differencing of a dozen pages taken at random. In some thousand values one large error was found, one obvious typing error and half a dozen errors in sign (in the neighborhood of the zeros). There are errors of one or two units in the end-figure in many places but this is expected as, in some cases, the authors have deliberately printed all the figures available from the computations, rather than sacrifice some for the sake of last-figure accuracy. As a war-time policy this is quite justifiable.

Strong criticism must be made of the awkward, and inconsistent, notations employed to denote very large or very small numbers. This is illustrated by the following examples:

Report No. 3. 0.626446 denotes $(6.26446) 10^{-6}$

Report No. 9. 2.2045 denotes $(2.2045) 10^4$

2.2045 denotes 2.2045 .

In the second case the notation is explained in the introduction but this can hardly excuse its use. In the cases when a blank space appears between the decimal point and the following figure it is particularly confusing.

An alternative method of spacing the rows of the tables might be preferred but, since there are not more than four columns to the page, no difficulty of interpretation arises.

4. Detailed review of the German tables.

Report No. 1.

The first report contains only odd values and is not of great interest.

Report No. 2.

The zeros j_m^n of J_n^2 are the same as the zeros j_m^n of J_n . These are tabulated to 7D for $m = 1(1)40$ in Watson, *Bessel Functions*, p. 748, and are accordingly omitted from the report.

The zeros were calculated using Newton's approximation with two correcting terms. The first approximations to the roots were obtained for m greater than or equal to 5 from McMahon's formula. For m less than 5, $n = 1$, they were found by inverse interpolation in Watson's table of J_1 . For m less than 5, $n = 2, 3$ or 4, the formula $2J_n' = J_{n-1} - J_{n+1}$ was used in conjunction with Jahnke & Emde's table of J_n .

The turning values were calculated from Taylor series. The last figures of both zeros and turning values are doubtful.

Reports Nos. 3, 5 and 8.

J_0, J_1, Y_0, Y_1 (which are denoted by N_0, N_1 as is usual in Germany), were taken from Watson's tables for values of the argument x up to 16. For x greater than 16 they were calculated from Hankel's asymptotic expansion, and the relation $J_1 N_0 - J_0 N_1 = 2/\pi x$ was used as a check.

For values of n less than x the recurrence formula was used with increasing n . For n greater than x the recurrence formula for the ratio J_{n-1}/J_n was used with decreasing n . Checks were made by comparison with Watson's values of $J_n(n)$. Since Watson's tables of J_0, J_1 , which form the basis of these tables, are to 7D the values given, also to 7D, are uncertain in the last figure. Subject to this uncertainty, seven figure accuracy can be obtained by using fourth differences, while linear interpolation gives three figure accuracy.

Report No. 8 also contains graphs of $J_n(x)$ and its first six derivatives for $n = 2, x = 0-28; n = 9, x = 5-33; n = 16, x = 10-38; n = 23, x = 20-48; n = 30, x = 25-53$. These were obtained from the identity between 2^n times the m th derivative of J_n and the m th difference, at unit interval, in the n direction.

Report No. 4.

The quantity $(n + \frac{1}{2}, m)$ is written in a form very convenient for computation for the range of values m to be covered;

$$(n + \frac{1}{2}, m) = N(N-1)(N-3) \dots (N-M)/m!$$

where

$$N = \frac{1}{2}n(n+1), \quad M = \frac{1}{2}m(m-1).$$

A check is provided by the relation $(n + \frac{1}{2}, n) = (2n)!/n!$

Reports Nos. 6 and 9.

The initial values of J_1 and J_{-1} were taken from Hayashi's tables of Bessel functions to six and more figures.

For positive values of the order, as in tables 3, 5 and 8, the two forms of the recurrence relation were used according as n exceeded or was less than x , using six decimals throughout. Checks were provided by differencing, and by values calculated from Hankel's and Debye's asymptotic series.

For negative values of the order the recurrence relation was used with increasing $|n|$ throughout the whole range, eight figures being retained. In the worst case the building up error is 300 times the rounding off error and it is accordingly possible to give the final values to 5S. Checks were provided by differencing, and by values calculated from Debye's asymptotic series.

Report No. 7.

This report contains remarks on various methods of computing Bessel functions and is based on the experience gained in the preparation of the tables given in the other reports. Comparisons are given of the results of various methods of computation, in particular of different forms of asymptotic series.

5. Unpublished British tables.

Some of the British Association Tables Committee's as yet unpublished work on Bessel functions is at present in the keeping of Admiralty Computing Service. Through the courtesy of the committee, values obtainable from these tables can be made available to establishments requiring them.

The tables consist of the four principal functions J , Y , I and K , or of auxiliary functions from which they can be deduced, roughly to 10 or 12 figures, for all integral values of the order up to 20, and at intervals of .1 in the argument up to 20. The list appended below, while not complete in every detail, is a sufficient indication of what values are available on request. Except where it is specifically stated otherwise, the interval of tabulation in the argument is .1 while the interval in the order is unity.

Function tabulated	Range of the order n	Range of the argument z	Number of decimals or significant figures
$J_n(x)$	2-14	0(.01)10	8D*
	0-20	0-25	10D*
	$-\frac{1}{2}, -\frac{3}{2}, +\frac{1}{2}, +\frac{3}{2}$	5-20	7D
$Y_n(x)$	0-21	0-21-25	13D and 19D
$y_n = x^n Y_n$	0-21	0-21	14S or 15S
$I_n(x)$	0-22	6-20	13D
$i_n = x^{-n} I_n$	0-22	0-6	15S
$e^{-x} I_n$	2-5	5-20	10S
	6-15	10-20	10S
	16-20	15-20	10S
	0-20	6-20	12S or 13S
$K_n(x)$	2-11	0(.01)5	8S*
$k_n = x^n K_n$	12-20	0-5	8S*
	6-15	5-10	10S
	16-20	5-15	10S
	0-7	5-20	9S or 10S
$e^x K_n$	8-15	10-20	9S or 10S
	16-20	15-20	9S or 10S

The tables of $J_n(x)$ and of $k_n(x)$ marked with asterisks are available in the form of bound photostats for loan for a limited period. It should be emphasized that these are not the Committee's final tables and that they are only made available, by courtesy of the Committee, in order not to hold up the work of establishments having a requirement for them.

Two other tables from which values can be supplied are those of the zeros of J_n and J'_n . j_{ns} is tabulated to 4D for $n = 0(1)19$, $s = 1(1)8$, and j'_{ns} to 4D for $n = 0(1)22$, $s = 1(1)9$.

MECHANICAL AIDS TO COMPUTATION

17[Z].—H. A. PETERSON & C. CONCORDIA, "Analyzers for use in engineering and scientific problems," *General Electric Review*, v. 48, September, 1945, p. 29-37. 20.7×28.5 cm.

This paper describes each of the following four analyzers in use by the General Electric Company:

- (1) The direct current network analyzer,
- (2) The alternating current network analyzer,
- (3) The transient network analyzer,
- (4) The differential analyzer.

A short account of the engineering applications of each analyzer is given, together with illustrations, and a bibliography of 63 titles, of which 14 of the 18 on Differential Analyzers were given in *MTAC*, v. 1, p. 452-454.

D. H. L.

NOTES

45. DAWSON'S OR POISSON'S INTEGRAL.—Various tables involving $\int_0^x e^{-t^2} dt$ have been already noted, *MTAC*, v. 1, p. 322-323, 422-423. H. G. Dawson's table, appearing originally in London Math. So., *Proc.*, v. 29, 1898, p. 521-522, was recomputed and reprinted with corrections by H. M. TERRILL & LUCILE SWEENEY, in Franklin Institute, *J.*, v. 238, Sept. 1944, p. 220f. An abridgement of Dawson's table is given in JAHNKE & EMDE, *Tables of Functions*, 1933 and later eds., $x = [0(.01)2; 4S]$. H. B. drew my attention to the fact that the integral here discussed occurs in Poisson's paper of 1815, "Sur la théorie des ondes," Acad. d. Sci., *Mémoires*, n.s., v. 1, 1818, p. 128-130. Two more references to tables involving the integral are as follows:

1. H. LAMB, "On water waves due to disturbance beneath the surface," London Math. So., *Proc.*, s. 2, v. 21, 1922, p. 367; table of $F(x) = x + (1 - 2x^2)e^{-x^2}\int_0^x e^{t^2} dt$, and of $F'(x)$, for $x = [0(.1)2; 3D]$. Also, p. 368, graph of $F(x)$, $0 < x \leq 2$.
2. Y. NOMURA, "On the waves of water of finite depth due to disturbance beneath the surface," Japan, Tôhoku teikoku daigaku, *Science Reports*, s. 1, math. phys. chem., v. 25, 1936, p. 1082; table of $\psi_0(x) = 2\int_0^x t e^{-t^2} \sin(2xt) dt = x + (1 - 2x^2)e^{-x^2}\int_0^x e^{t^2} dt$, $x = [0(.1)2(1)6-(2)10; 4D]$. Also, for the same range, tables of

$$\psi_s(x) = \sum_{r=0}^s \frac{x^{2r+1}}{(s-r)!r!} \frac{d^s}{d(x^2)^s} [x^{2s-2r-1}\psi_0(x)], \quad s = 1, 2, 3.$$

R. C. A.

46. FOUR-POINT LAGRANGEAN INTERPOLATION COEFFICIENTS FOR UNUSUAL FRACTIONS OF THE INTERVAL.—In QR 20 it is suggested that $\tan^{-1}(m/n)$ could be obtained by interpolation in the NYMTP *Table of Arc Tan x*, 1942. A suitable formula is the four-point Lagrange formula, which may be written

$$f_p = A_{-1}f_{-1} + A_0f_0 + A_1f_1 + A_2f_2.$$

In this f_p is written for $f(a + pw)$, where w is the tabular interval. Since

$$A_{-1}(p) = A_2(1 - p) = A_2(q), \quad A_0(p) = A_1(1 - p) = A_1(q)$$

where $q = 1 - p$, we have also

$$f_q = A_2f_{-1} + A_1f_0 + A_0f_1 + A_{-1}f_2,$$

so that we need a table of coefficients only for $0 \leq p \leq 1/2$.

Tables of Lagrange coefficients, when $10p$ or $100p$ is an integer, are common; the NYMTP *Tables of Lagrangian Interpolation Coefficients*, 1944, gives the coefficients to 10D when $10\,000p$ is an integer. These tables also give the coefficients in fractional form when $12p$ is an integer.

The accompanying Tables give the coefficients when $14p$, $18p$, $22p$, $26p$ or $30p$ is an integer. These coefficients are rarely needed, and no published table is known to the writer. They have been checked by application to the functions $f(x) = 1$ and $f(x) = 2x - 1$.

p	DA ₋₁	DA ₀	DA ₁	DA ₂	q	D
1/14	- 351	+ 15795	+ 1215	- 195	13/14	16464
1/7	- 39	+ 936	+ 156	- 24	6/7	1029
3/14	- 825	+ 14025	+ 3825	- 561	11/14	16464
2/7	- 60	+ 810	+ 324	- 45	5/7	1029
5/14	- 1035	+ 11799	+ 6555	- 855	9/14	16464
3/7	- 66	+ 660	+ 495	- 60	4/7	1029
1/2	- 1	+ 9	+ 9	- 1	1/2	16
1/18	- 595	+ 33915	+ 1995	- 323	17/18	34992
1/9	- 68	+ 2040	+ 255	- 40	8/9	2187
1/6	- 55	+ 1155	+ 231	- 35	5/6	1296
2/9	- 112	+ 1848	+ 528	- 77	7/9	2187
5/18	- 2015	+ 27807	+ 10695	- 1495	13/18	34992
1/3	- 5	+ 60	+ 30	- 4	2/3	81
7/18	- 2233	+ 23925	+ 15225	- 1925	11/18	34992
4/9	- 140	+ 1365	+ 1092	- 130	5/9	2187
1/2	- 1	+ 9	+ 9	- 1	1/2	16
1/22	- 903	+ 62307	+ 2967	- 483	21/22	63888
1/11	- 105	+ 3780	+ 378	- 60	10/11	3993
3/22	- 2337	+ 58425	+ 9225	- 1425	19/22	63888
2/11	- 180	+ 3510	+ 780	- 117	9/11	3993
5/22	- 3315	+ 53703	+ 15795	- 2295	17/22	63888
3/11	- 228	+ 3192	+ 1197	- 168	8/11	3993
7/22	- 3885	+ 48285	+ 22533	- 3045	15/22	63888
4/11	- 252	+ 2835	+ 1620	- 210	7/11	3993
9/22	- 4095	+ 42315	+ 29295	- 3627	13/22	63888
5/11	- 255	+ 2448	+ 2040	- 240	6/11	3993
1/2	- 1	+ 9	+ 9	- 1	1/2	16

p	DA ₋₁	DA ₀	DA ₁	DA ₂	q	D
1/26	- 1275	+ 103275	+ 4131	- 675	25/26	105456
1/13	- 150	+ 6300	+ 525	- 84	12/13	6591
3/26	- 3381	+ 98049	+ 12789	- 2001	23/26	105456
2/13	- 264	+ 5940	+ 1080	- 165	11/13	6591
5/26	- 4935	+ 91791	+ 21855	- 3255	21/26	105456
3/13	- 345	+ 5520	+ 1656	- 240	10/13	6591
7/26	- 5985	+ 84645	+ 31185	- 4389	19/26	105456
4/13	- 396	+ 5049	+ 2244	- 306	9/13	6591
9/26	- 6579	+ 76755	+ 40635	- 5355	17/26	105456
5/13	- 420	+ 4536	+ 2835	- 360	8/13	6591
11/26	- 6765	+ 68265	+ 50061	- 6105	15/26	105456
6/13	- 420	+ 3990	+ 3420	- 399	7/13	6591
1/2	- 1	+ 9	+ 9	- 1	1/2	16
1/30	- 1711	+ 159123	+ 5487	- 899	29/30	162000
1/15	- 203	+ 9744	+ 696	- 112	14/15	10125
1/10	- 57	+ 1881	+ 209	- 33	9/10	2000
2/15	- 364	+ 9282	+ 1428	- 221	13/15	10125
1/6	- 55	+ 1155	+ 231	- 35	5/6	1296
1/5	- 6	+ 108	+ 27	- 4	4/5	125
7/30	- 8533	+ 135309	+ 41181	- 5957	23/30	162000
4/15	- 572	+ 8151	+ 2964	- 418	11/15	10125
3/10	- 119	+ 1547	+ 663	- 91	7/10	2000
1/3	- 5	+ 60	+ 30	- 4	2/3	81
11/30	- 10241	+ 114513	+ 66297	- 8569	19/30	162000
2/5	- 8	+ 84	+ 56	- 7	3/5	125
13/30	- 10387	+ 103071	+ 78819	- 9503	17/30	162000
7/15	- 644	+ 6072	+ 5313	- 616	8/15	10125
1/2	- 1	+ 9	+ 9	- 1	1/2	16

J. C. P. MILLER

Advocates

47. THE GRAEFFE PROCESS.—~~Protagonists~~ of the root-squaring method¹ for evaluating the roots of an algebraic polynomial equation usually claim that it renders possible the determination, in one process, of all the roots, both real and complex. This assertion is not true of the method as ordinarily expounded, but Lehmer's criticism,² that the process fails utterly in such simple cases as the equation

$$(1) \quad x^4 + x^3 + x^2 + x + 1 = 0,$$

seems unduly harsh. Applying the Graeffe algorithm, (1) reproduces itself; hence the square of any root of (1) is also a root, all the roots are of modulus unity, and it is not a big step thence to deduce the roots in the forms

$$x = \cos 2\pi/5 \pm i \sin 2\pi/5, \cos 4\pi/5 \pm i \sin 4\pi/5.$$

Numerical values can be calculated by Runge's method,³ applying the Graeffe algorithm to a modified equation obtained from (1) by adding a constant to the roots.

The simple Graeffe algorithm, applied to any algebraic polynomial equation whatever, separates the roots into groups having distinct⁴ moduli, and determines the moduli, whatever their multiplicities. The real parts of

all roots whose moduli are simple or double can be determined without ambiguity by using the additional algorithm of Brodetsky & Smeal.⁵ These statements together imply that almost every practical equation can be completely solved by the combined algorithm; in those exceptional cases where moduli of higher multiplicity are found, however, the complete determination of the roots requires a further calculation with a different technique. The method of Brodetsky & Smeal uses an infinitesimal change of origin, and rests on the assumption that this change will not alter the way in which the equation breaks up under repeated application of the Graeffe algorithm. This assumption is not justified when roots exist having the same modulus but different real parts. Runge's method is still available, however, and can determine the roots without ambiguity. The factors corresponding to the completely determined roots should first be removed, and the change of origin then made: if there is more than one modulus of multiplicity higher than double, ambiguity is still avoidable by choosing the change of origin less than the least difference between the moduli concerned.

It remains to consider the case where roots exist having moduli so nearly equal that separation by the ordinary Graeffe process is too tedious to be practicable. Again it will be best to remove the factors corresponding to the other roots, leaving a residual equation to which we apply the algorithms of Graeffe, and Brodetsky & Smeal, after a change of origin.⁶ In general, separation will be effected in this way, and the roots of the proposed equation can be inferred from those of the modified equation. If, on the other hand, separation of the modified equation is still impracticable, then the roots are clearly nearly equal (in conjugate pairs, if complex). Knowledge of the approximate moduli with or without the shift of origin enables us, as in the normal Runge process, to infer the approximate value of the roots. A high degree of accuracy is not to be expected, in view of the poorly determinate nature of nearly equal roots.⁷

This brief discussion shows that Graeffe's method, with the auxiliary processes which have been devised to determine the real parts of the complex roots, is competent to solve any algebraic polynomial equation whatever. If objections are to be raised against the method, it must not be on account of its lack of generality, but mainly on account of the fact that it is neither self-correcting nor self-checking. If one makes an error in the numerical work, then essentially the wrong equation is solved. This is a weakness in any computing process, but when the process is of a nature that seems to invite mistakes (and the wanderings of the decimal point under the Graeffe algorithm are most error-provoking), it constitutes a serious defect. A further defect is the difficulty of assessing the accuracy of the roots obtained: an independent checking and correcting process is desirable. For these and other reasons, the author has found that iterative methods of evaluating the roots are much to be preferred, save where root location is a matter of difficulty. A paper on some new and powerful iteration methods is being prepared; in the interim the methods given by SHIHLINGE LIN⁸ are often all that is needed.

K. MITCHELL

King's College, Newcastle upon Tyne

¹ See E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, third ed., London, 1940.

² D. H. LEHMER, "The Graeffe process as applied to power series," *MTAC*, v. 1, p. 377f.

³ C. RUNGE & H. KÖNIG, *Vorlesungen über numerisches Rechnen*, Berlin, 1924. The method was given earlier in C. RUNGE, *Praxis der Gleichungen (Sammlung Schubert)*, Leipzig, 1900. [EDITORIAL NOTE: There was a second improved edition of this work (*Götschen's Lehrbücherei*, v. 2), Berlin, 1921.]

⁴ There is a practical limitation here, to which we return below. The moduli must be sufficiently distinct to separate in a reasonable number of steps: regarding 8 as the reasonable limit, moduli in the ratios 1.0072 or 1.0036 to 1 will be separated, to 1 part in 10^4 or 10^5 respectively.

⁵ S. BRODETSKY & G. SMEAL, "On Graeffe's method for complex roots of algebraic equations," *Camb. Phil. So., Proc.*, v. 22, 1924, p. 83f.

⁶ A shift of origin through a distance about equal to the approximate value of the modulus is recommended as certain to separate the roots, if not actually nearly equal.

⁷ A. OSTROWSKI, "Sur la continuité relative des racines d'équations algébriques," *Académie d. Sci., Paris, Comptes Rendus*, v. 209, 1939, p. 777f, has illustrated this very forcibly by comparison of $z^4 - 4z^3 + 6z^2 - 4z + 1 = 0$, roots 1, 1, 1, 1; with $z^4 - 4z^3 + 5.999951z^2 - 4z + 1 = 0$, roots 1.0872, .9198, .9965 \pm .0836i.

⁸ SHIH-NGE LIN, "A method for finding roots of algebraic equations," *J. Math. Phys.*, v. 22, 1943, p. 60f.

48. GUIDE (*MTAC*, no. 7), SUPPL. 2 (for Suppl. 1, see *MTAC*, v. 1, p. 403-404).—In L. SILBERSTEIN, *Bell's Mathematical Tables*, London, 1922, or *Synopsis of Applicable Mathematics with Tables*, New York, 1923, there are tables of $J_0(x)$, $J_1(x)$, $x = [0(.01)15.5; 5D]$, p. 143-150, and $K_0(x)$, $x = [0(.01)1(.1)9.9; 4D]$, p. 152.

G. GIORGI, "Sugli integrali dell'equazione di propagazione in una dimensione," *Circolo Matem. d. Palermo, Rendiconti*, v. 52, 1928, p. 311-312; tables of $I_1(x)$, $x = [0(.1)6(1)11; 4-6S]$, $B_n(x) = I_1(x)e^{-x}/x$, $x = [0(.1)6(.5)-9(1)20(5)30(10)100(100)500, 1000; 4-6D]$.

In R. C. KNIGHT, "The potential of a sphere inside an infinite circular cylinder," *Q. J. Math.*, v. 7, 1936, p. 130, there is a table of $I_{2n} = \int_0^\infty K_0(m)m^{2n}dm/I_0(m)$, $n = [0(1)6; 5D$ or $S]$. There are also tables for the same integral, but with the limits 0 to 1, 1 to 5, and 5 to ∞ .

R. ZURMÜHL, "Zur numerischen Integration gewöhnlicher Differentialgleichungen zweiter und höherer Ordnung," *Z. angew. Math. Mech.*, v. 20, 1940, p. 116; tables of $y = \int_0^x e^{-x \sin t} dt = \frac{1}{2}\pi[I_0(x) - L_0(x)]$, and $-y' = 1 - \frac{1}{2}\pi[I_1(x) - L_1(x)]$, $x = [0(.1)2(.2)10; 7-8D]$. Errors up to 2 units in the last decimal possible. $L_0(x) = -iH_0(ix)$ (Watson, p. 329). See also R. Müller, *Z. angew. Math. Mech.*, v. 19, 1939, p. 54, where there are tables of y and $-y'$, $x = [0(.1)1(.2)13.6(.4)16; 3-5D]$. On p. 53 Müller gives also a table of $K_0(x)$, $x = [0, .02, .04, .1(.1)2.6(.2)13.6(.4)16; 6-9S]$, δ^2 .

R. C. A.

49. A VOLUME OF TABLES BY KULIK.—Brown University has recently acquired a copy of tables by JAKOB PHILIPP KULIK (1793-1863), entitled *Handbuch mathematischer Tafeln*, and published by Christoph Penz at Graz, in 1824. liv, 149 p. + 1 p. Druckfehler. 13.8×20 cm. The book came from the fine library of CHARLES N. HASKINS of Dartmouth College. This volume is not listed by DE MORGAN (1861), GLAISHER (1873), HENDERSON (1926), LEHMER (1941), Wölffing (1903), or in catalogues of Bibliothèque Nationale (1925), British Museum (1890), Crawford Library of the Edinburgh Univ. (1890), Hamburg Math. So. Lib. (1890, 1894, 1906, 1913), Library of Congress (1944), Pulkowa Observatory (1860, 1880), R. Astron.

So. (1886, 1900), Stadtbibliothek of Frankfort a. M., math. Abt. (1909). In only two sources did I find the work listed, namely: in POGGENDORFF, *Biog.-liter. Handwörterbuch*, v. 1, Leipzig, 1863, a contribution from Kulik; and in D. BIERENS DE HAAN, *Akad. v. Wetenschappen*, Amsterdam, *Verhandelungen*, v. 15, 1875, "Tweede Ontwerp eener Naamlijst van Logarithmentafels."

In the "Vorerinnerung," dated November, 1823, Kulik states that his Tafeln are an extract from a larger work, to be published in the year 1824, and entitled *Collectio tabularum mathematico-physicarum locupletissima, vollständige Sammlung mathematisch-physicalischer Tafeln*. Among various volumes of tables which Kulik wrote I do not find that this one is ever mentioned, not even in the bibliography sent by Kulik to Poggendorff (1863) nearly 40 years later. But he did publish at Graz, in 1825, the following volume of 266 p.: *Divisores numerorum decies centena millia non excedentium. Accedunt tabulae auxiliares ad calculandos numeri cujuscunque divisores destinae. Tafeln der einfachern Factoren aller Zahlen unter einer Million nebst Hilfstafeln zur Bestimmung der Factoren jeder grösseren Zahl*. This v. is mentioned in none of the bibliographies listed above, except Poggendorff (1863), but it is surveyed in BAASMTTC, *volume V, Factor Table*, London, 1935, p. xiii.

About a decade ago L. J. C. drew my attention to *Astron. Nachrichten*, v. 3, 1825, col. 192, where Kulik describes a work with the following title: *Canon logarithmorum naturalium in 48 notis decimalibus pro omnibus numeris inter 1 et 11000 denuo in computum vocatus ab Jac. Phil. Kulik*. He stated (v. 4, 1826, col. 47) that 192 of the projected 288 pages had already been printed but that the printing of further pages depended upon securing subscribers. The book seems never to have been completed. Is there any library which has these 192 pages? See *Scripta Mathematica*, v. 4, 1936, p. 340.

In the *Handbuch* are 30 tables including the following:

- 1-2: All factors of numbers up to 21500 and the smallest factors up to 67100.
- 3-4: Squares and cubes of numbers up to 1000 and higher powers of numbers up to 100.
- 5: Square roots and cube roots of numbers up to 1000.
- 15, 19: Natural and logarithmic sin and tan.
- 16: Natural secants.
- 28: 11-place log of prime numbers up to 1811.

R. C. A.

50. ZEROS OF $z + \sin z$.—The zeros of $z + \sin z$ are situated symmetrically in the four quadrants. The first four zeros in the first quadrant were found as follows. A first approximation to the imaginary part, y , of the n^{th} zero is

$$y = \ln(4n - 1)\pi.$$

The imaginary part is a solution of

$$(\sinh^2 y - y^2)^{\frac{1}{2}} \coth y - \arccos(\sinh y) = 0$$

and was obtained by inverse interpolation. The real part of the zero is then given by

$$x = \arccos(-y/\sinh y).$$

Zeros of $z + \sin z$ where $z = x + iy$

n	$\pm x$	$\pm y$
1	4.2123922	2.2507286
2	10.7125374	3.1031487
3	17.0733649	3.5510873
4	23.3983552	3.8588090.

If one considers the roots of $\sin z = z$ as functions of n and interpolates for the roots corresponding to $n = 4\frac{1}{2}, 5\frac{1}{2}, \dots, 9\frac{1}{2}$, one obtains the zeros of $z + \sin z$ for $n = 5, 6, \dots, 10$. Using the first ten roots of $\sin z = z$ as given in an earlier article by HILLMAN & SALZER,¹ the above mentioned roots of $z + \sin z$ can be obtained to at least four decimal places.

NYMPT

BEATRICE S. MITTELMAN & A. P. HILLMAN

¹ A. P. HILLMAN & H. E. SALZER, "Roots of $\sin z = z$," *Phil. Mag.*, s. 7, v. 34, 1943, p. 575. See *MTAC*, v. 1, p. 141.

EDITORIAL NOTE.—In *Ingenieur-Archiv*, v. 11, 1940, p. 129, J. FADLE gave the first five zeros of $\sin z \pm z$, to 5D. Comparing with the seven-place values listed above, it appears that six last-figure endings of Fadle should be increased by unity, namely in the real parts of the second and fourth zeros, and in the imaginary parts of the first four zeros. Comparing Fadle's zeros of $\sin z - z$ to 5D with those found by Hillman & Salzer¹ to 6D, we find that all of Fadle's end-figures in the first three zeros, as well as the end-figure in the real part of the fourth zero, should be increased by unity.

QUERIES

16. TABLES OF $\sin nx/\sin x$.—Has any table been calculated for the function $\sin nx/\sin x$, for large integral values of n , say up to 100, and for values of x in radians?

DOROTHY M. WRINCH

Smith College,
Northampton, Mass.

EDITORIAL NOTE: In NYMTP, *Tables of Sines and Cosines for Radian Arguments*, 1940, are values of $\sin x$, $x = [0(1)100; 8D]$.

QUERIES—REPLIES

19. CUBE ROOTS (Q 11, v. 1, p. 372; QR 15, v. 1, p. 432). The answer to Q 11 seems to be a definite 'No.' The table required is equivalent to one giving 5 or 6D for $N = 1000(0.1)2000$, and a table at 10 times this interval is already interpolable linearly, so that a printed table at interval 0.1, although it might be very convenient, cannot be considered an urgent need. For a table at interval 1, the 1930 and 1941 editions of *Barlow's Tables* seem most convenient.

Use of linear interpolation when the second difference is about 2 units means, of course, that the last figure is subject to a maximum error of about $1\frac{1}{2}$ units, whereas tabular values are usually kept within half a unit.

If a full unit is not allowable, then an extra decimal must be kept; in this case linear interpolation may be inaccurate to the extent of 2 or 3 units in the extra decimal, and second differences may have to be used. If this is the case, the method of QR 15 is probably easier for isolated values, but for a succession of values with exact one-decimal arguments, subtabulation seems indicated, for example by the end-figure process described in the *Nautical Almanac* for 1931.

J. C. P. MILLER

20. TABLES OF $\tan^{-1}(m/n)$ (Q 14, p. 431; QR 18, p. 460).—I know of no published table of $\tan^{-1}(m/n)$, but I am preparing one of this type as a preliminary to the preparation of a table of $\ln \Gamma(x + iy)$ with $x = 1/4, 1/2, 3/4$ for $y = 0.(1)2$ and beyond (compare RMT 234). The need arises when applying the relation

$$\begin{aligned}\ln \Gamma(x + iy + 1) &= \ln(x + iy) + \ln \Gamma(x + iy) \\ &= \frac{1}{2} \ln(x^2 + y^2) + i \tan^{-1}(y/x) + \ln \Gamma(x + iy)\end{aligned}$$

in order to transfer the argument to a region where an expansion in series converges with sufficient rapidity. So far, my table gives $\tan^{-1}(m/n)$ for $n = 10(10)110$ and $m = [0(1)50; 12D]$, or, as it may be written, for $n = 1(1)11, m = 0.(1)5$. It is now being extended by Mr. S. Johnston.

An obvious way of producing or extending such a table is to interpolate in the NYMTP *Table of Arc Tan x*, 1942. This may be done by using the four-point Lagrange formula and it has seemed worth while to tabulate the coefficients when the interval is divided into 14, 18, 22, 26 or 30 parts. These are given in N 46.

Nevertheless, interpolation in the table can be avoided and it may be of interest to others to have a description of the method I have used.

Firstly, we may note that

$$\tan^{-1}(m/n) = \frac{1}{2}\pi - \tan^{-1}(n/m)$$

so that the NYMTP tables may be used without interpolation whenever either m/n or n/m is a tabular argument. This accounts for almost all cases that normally arise except those in which m and n contain different primes (exclusive of 2 and 5) as factors.

Secondly, and more systematically, we may use relations of the type

$$\begin{aligned}\tan^{-1}(\lambda + 1/3) &= \tan^{-1} \lambda + \tan^{-1} 1/(3\lambda^2 + \lambda + 3) \\ &= \frac{1}{2}\pi + \tan^{-1} \lambda - \tan^{-1}(3\lambda^2 + \lambda + 3)\end{aligned}$$

which expresses $\tan^{-1}(\lambda + 1/3)$, for example when λ is an integer (not too large), in terms of two tabular entries. Likewise, we have

$$\tan^{-1}(\lambda - 1/3) = -\pi/2 + \tan^{-1} \lambda + \tan^{-1}(3\lambda^2 - \lambda + 3).$$

Thus

$$\begin{aligned}\tan^{-1} 1/3 &= \pi/2 - \tan^{-1} 3 &= 0.32175 \ 05543 \ 97 \\ \tan^{-1} 2/3 &= -\pi/4 + \tan^{-1} 5 &= 0.58800 \ 26035 \ 48 \\ \tan^{-1} 4/3 &= 3\pi/4 - \tan^{-1} 7 &= 0.92729 \ 52180 \ 01 \\ \tan^{-1} 5/3 &= -\pi/2 + \tan^{-1} 2 + \tan^{-1} 13 &= 1.03037 \ 68265 \ 24\end{aligned}$$

and so on.

With the NYMTP tables λ need not be an integer; multiples of 0.1, for example, will also serve. We may also use relations such as

$$\tan^{-1}(\lambda + 10/3) = \frac{1}{2}\pi + \tan^{-1}\lambda - \tan^{-1}(3\lambda^2 + 10\lambda + 3)/10.$$

Such relations have been found useful when $3\lambda^2 + \lambda + 3$ tends to be too big to be a tabular argument.

In some individual cases, it was found necessary to search rather carefully for suitable expressions, and in these circumstances, no saving of time results when comparison is made with straightforward methods of interpolation. It may be of interest, however, to note a few of these special cases,¹ suitable for use with the NYMTP tables:

$$\begin{aligned}\tan^{-1}(4.9/9) &= \tan^{-1}0.5 - \tan^{-1}13 + \tan^{-1}23.88 \\ \tan^{-1}(4.7/9) &= \tan^{-1}0.5 - \tan^{-1}25.5 + \tan^{-1}46.34 \\ \tan^{-1}(3.1/6) &= \frac{1}{2}\pi + \tan^{-1}0.5 - \tan^{-1}75.5 \\ \tan^{-1}(3.7/6) &= \pi - \tan^{-1}1.625 - \tan^{-1}1076 \\ \tan^{-1}(4.1/7) &= -\frac{1}{2}\pi + \tan^{-1}0.6 + \tan^{-1}94.6\end{aligned}$$

The first two of these cases are the only ones I have found so far which seem to need three tabular values; the others are typical of a fairly large number needing two tabular values together with a multiple of π . As a rule, I tried to express each value required in terms of two values only, one being a tabular value and the other a multiple of π if possible or, if not, a tabular value.²

J. C. P. MILLER

¹ It has been found that

$$\begin{aligned}\tan^{-1}(4.9/9) &= -\frac{1}{2}\pi + \tan^{-1}0.6 + \tan^{-1}23.88 \text{ (S.A.J.)}, \\ \tan^{-1}(4.7/9) &= -\frac{1}{2}\pi + \tan^{-1}0.55 + \tan^{-1}46.34 \text{ (S.A.J.)}, \\ &= \frac{1}{2}\pi + \tan^{-1}0.3 - \tan^{-1}5.205 \text{ (J.C.P.M.)};\end{aligned}$$

we may also note that

$$\tan^{-1}(3.7/6) = \frac{1}{2}\pi + \tan^{-1}0.6 - \tan^{-1}82.2 \text{ (S.A.J.)}.$$

² Considerations in connection with this communication have suggested the question: What values of $\tan^{-1}N$ (N being an integer) are fundamental? For instance,

$$\begin{aligned}\tan^{-1}3 &= 3\tan^{-1}1 - \tan^{-1}2 \\ \tan^{-1}7 &= 2\tan^{-1}2 - \tan^{-1}1 \\ \tan^{-1}8 &= 2\tan^{-1}1 + \tan^{-1}3 - \tan^{-1}5 \\ \tan^{-1}13 &= 5\tan^{-1}1 - \tan^{-1}2 - \tan^{-1}4\end{aligned}$$

So we certainly do not need $N = 3, 7, 8, 13$. But do we need all of 1, 2, 4, 5, 6, 9, 10, 11, 12?

CORRIGENDA

P. 54, l. -2, for e^{100} (117D), read e^{100} (116S); l. -3, for e^8 (255D), read e^8 (254D).

P. 55, l. 18, for $\ln 17$ to 224D, read $\ln 17$ to 274D.

P. 56, l. 1, for Recalculation of the modulus, read Recalculation and extension of the modulus; l. 16, for e^{10} to 289D, read e^{10} to 284D.

P. 59, l. -21, for The correct value, read The value.

P. 227, delete A₃ 5, 13.

P. 229, delete B₃ 3, 4.

P. 253, l. -4, for 2 $\frac{1}{2}$, read 2 $\frac{1}{4}$; for $\text{ber}'x$, read $-\text{ber}'x$; l. -2, for $\text{ker}'x$, read $-\text{ker}'x$; for kei_2x , read $-\text{kei}_2x$.

- P. 283, BAASMTTC 4, 5, for $I_n(x)$ and $i_n(x) = x^{-n}I_n(x)$, and $e^{-x}I_n(x) \cdots n = 2(1)12$, read $i_n(x) = x^{-n}I_n(x)$, or $e^{-x}I_n(x) \cdots n = 2(1)11$; for $K_n(x)$, and $k_n(x) = x^n K_n(x)$, and $e^x K_n(x) \cdots n = 2(1)12$, read $k_n(x) = x^n K_n(x)$, or $e^x K_n(x) \cdots n = 2(1)11$.
- P. 289, FRESNEL 1₂, for "apparently calculated without knowledge of Fresnel's table," read "exactly Fresnel's table with its errors"; under 1₂ delete "thus, while some improvements were made, very little is added to Fresnel's table."
- P. 294, KARAS, l. 1, for 1937, read 1936.
- P. 304, B. A. SMITH, l. 1, for 1926, read 1896.
- P. 305, 432, 478, for J. STEINER, read L. Steiner; and p. 305, under H. STRUVE, l. 1, 4, for s. 2, read s. 3.
- P. 415, l. -12, for *Proc.*, read *Trans.*
- P. 449, delete line 14.
- P. 456, l. 2, for vanish., read vanish."

RMT 233 Addendum, p. 13

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 - C. Logarithms
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 - E. Hyperbolic and Exponential Functions
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